Worst-Case Analysis, Safety Margins, and Fuzzy Algebra: A Mathematical Equivalence

With applications to truss structures and multivalued logic

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Abstract

This report establishes a rigorous mathematical equivalence between three areas that appear, at first glance, to be unrelated: classical structural safety analysis based on allowable stresses and safety factors, fuzzy set theory and its α-cut formalism, and multivalued logic.

The central result is that the *safety margin* $\mathrm{SM}_i(P_1, P_2)$ of every structural member is, precisely, a fuzzy membership function mapping the load plane into $[0, 1]$. The α-cut condition $\mathrm{SM}_i \geq \alpha$ recovers exactly the allowable-force domain corresponding to a prescribed safety factor. Global worst-case analysis of the whole structure is governed by

$$\mathrm{SM}(P_1, P_2) = \min\{\mathrm{SM}_1(P_1, P_2), \ldots, \mathrm{SM}_m(P_1, P_2)\},$$

which is the standard fuzzy-intersection formula.

Propagating loads through the structural model via axial-force equilibrium is shown to be exactly Zadeh's extension principle. The dependency problem that arises when safety margins are re-expressed in terms of internal forces is identified and explained through the lens of interval arithmetic.

Finally, the function $\mu(P_1, P_2) = \mathrm{SM}(P_1, P_2)$ provides a fully interpretable, physically motivated definition of the membership grade as a multivalued degree of truth. The framework is illustrated on a five-bar plane truss subject to two independent loads.

Contents

1 Introduction

Engineering design uses deterministic safety criteria every day. A structural element is deemed acceptable if its internal stress remains below a prescribed allowable value; a safety factor specifies by how much the capacity exceeds the demand. These ideas are centuries old and are codified in building codes worldwide [14, 15].

Fuzzy set theory, introduced by Zadeh [2], provides a mathematical language for gradual transitions: an element does not simply pass or fail — it belongs to the "safe" set to some *degree*. The extension principle [3, 4] allows fuzzy quantities to be propagated through arbitrary functional relationships.

At the same time, the classical many-valued logics of Łukasiewicz [7] and Post [8] replace the Boolean truth values $\{0, 1\}$ with the unit interval $[0, 1]$, interpreting intermediate values as partial or graded truth.

The purpose of this report is to show that these three threads — structural safety, fuzzy algebra, and multivalued logic — are not merely analogous but *mathematically identical* in the context of load-carrying structures. No probabilistic or possibilistic assumptions are required; the equivalence is a purely algebraic consequence of the safety margin definition.

The development is self-contained. Section 2 defines the safety margin and connects it to safety factors. Section 3 identifies the safety margin as a fuzzy membership function and derives the alpha-cut equivalence. Section 4 establishes the global worst-case / fuzzy-intersection formula. Section 5 works through a five-bar plane truss in detail. Section 6 shows how the extension principle arises from load-to-force propagation. Section 7 analyses the dependency problem. Section 8 completes the circle by identifying the safety margin as a degree of truth in the sense of multivalued logic.

2 Safety Margins and Safety Factors in Structural Mechanics

2.1 Stress and allowable stress

Consider a prismatic bar (member i) with cross-sectional area $A_i > 0$ carrying an axial internal force N_i. The axial stress is

$$\sigma_i = \frac{|N_i|}{A_i}.$$

Let $\sigma_{\max} > 0$ be the allowable stress, a material and design-code dependent constant [16, 17]. The classical adequacy criterion is $\sigma_i \leq \sigma_{\max}$.

2.2 Safety factor

Safety Factor

The *safety factor* of member i is

$$\mathrm{SF}_i = \frac{\sigma_{\max}}{\sigma_i} = \frac{A_i\,\sigma_{\max}}{|N_i|}.$$

The element satisfies the design criterion if and only if $\mathrm{SF}_i \geq \mathrm{SF}_{\mathrm{req}}$ for a specified required value $\mathrm{SF}_{\mathrm{req}} \geq 1$.

Typical required safety factors range from 1.5 (steel, dead loads) to 3 or higher for brittle materials or dynamic loading [15].

2.3 Safety margin

Safety Margin

The *safety margin* of member i is the normalised reserve capacity

$$\mathrm{SM}_i = \max\!\left(0,\ 1 - \frac{|N_i|}{A_i\,\sigma_{\max}}\right) = \max\!\left(0,\ 1 - \frac{\sigma_i}{\sigma_{\max}}\right).$$

By construction $\mathrm{SM}_i \in [0, 1]$, with the following physical interpretation:

- $\mathrm{SM}_i = 1$ — the bar is completely unloaded;
- $0 < \mathrm{SM}_i < 1$ — the bar is stressed but within the allowable range;
- $\mathrm{SM}_i = 0$ — the allowable stress is reached or exceeded.

2.4 Algebraic relation between SM and SF

A direct calculation shows

$$\mathrm{SF}_i = \frac{1}{1-\mathrm{SM}_i} \qquad \text{and} \qquad \mathrm{SM}_i = 1 - \frac{1}{\mathrm{SF}_i}.$$

The two quantities therefore carry identical information. Moreover, the design condition $\mathrm{SF}_i \geq \mathrm{SF}_{\mathrm{req}}$ is equivalent to

$$\mathrm{SM}_i \;\geq\; \alpha_{\mathrm{req}}, \qquad \alpha_{\mathrm{req}} = 1 - \frac{1}{\mathrm{SF}_{\mathrm{req}}}.$$

This algebraic identity is the key bridge to fuzzy set theory developed in the next section.

3 Safety Margins as Fuzzy Membership Functions

3.1 Dependence on loads

Internal forces are determined by the applied loads $P = (P_1, P_2, \ldots, P_n)$ through the equilibrium equations of the structure. Write the resulting functional dependence as

$$N_i = N_i(P_1, P_2, \ldots, P_n).$$

Substituting into Definition 2.3 gives the *load-parameterised safety margin*

$$\mathrm{SM}_i(P_1, P_2, \ldots, P_n) \;=\; \max\!\Big(0,\; 1 - \frac{|N_i(P_1, \ldots, P_n)|}{A_i\,\sigma_{\max}}\Big). \tag{1}$$

3.2 Identification with a fuzzy membership function

Fuzzy Membership Function [ ]

A *fuzzy set* $\tilde{A}$ on a universe X is defined by a membership function $\mu_{\tilde{A}} : X \to [0,1]$, where $\mu_{\tilde{A}}(x)$ is the *degree of membership* of element x in $\tilde{A}$.

Safety Margin as Membership Function

The function $\mathrm{SM}_i : \mathbb{R}^n \to [0,1]$ defined by (1) satisfies all axioms of a fuzzy membership function on the load space $\mathbb{R}^n$. Under the identification

$$\mu_i(P_1, \ldots, P_n) \;=\; \mathrm{SM}_i(P_1, \ldots, P_n),$$

the fuzzy set $\tilde{S}_i$ represents the collection of load combinations for which member i is *safe to degree* SM_i.

Remark

No assumptions about uncertainty, imprecision, or random variation are required. The membership function arises from a purely deterministic stress formula; fuzzy algebra provides the correct mathematical language for describing its properties.

3.3 Alpha-cuts and allowable forces

α-Cut [,]

For $\alpha \in [0,1]$, the α-cut of a fuzzy set $\tilde{A}$ is

$$[\tilde{A}]_\alpha = \{x \mid \mu_{\tilde{A}}(x) \geq \alpha\}.$$

α-Cut $\equiv$ Allowable Load Domain

The α-cut of the safety fuzzy set $\tilde{S}_i$ is

$$[\tilde{S}_i]_\alpha = \{(P_1, \ldots, P_n) \mid \mathrm{SM}_i(P_1, \ldots, P_n) \geq \alpha\} = \{(P_1, \ldots, P_n) \mid |N_i(P)| \leq (1-\alpha)\, A_i\, \sigma_{\max}\}.$$

This is exactly the domain of loads for which the allowable internal force does not exceed $(1-\alpha)\, A_i\, \sigma_{\max}$.

Proof. $\mathrm{SM}_i(P) \geq \alpha$ iff $1 - |N_i(P)|/(A_i \sigma_{\max}) \geq \alpha$ iff $|N_i(P)| \leq (1-\alpha) A_i \sigma_{\max}$. □

Safety Factor as α-Level

Combining Theorem 3.3 with the relation $\alpha_{\mathrm{req}} = 1 - 1/\mathrm{SF}_{\mathrm{req}}$ yields:

Choosing a required safety factor $\mathrm{SF}_{\mathrm{req}}$ *is mathematically equivalent to selecting the α-cut level* $\alpha_{\mathrm{req}} = 1 - 1/\mathrm{SF}_{\mathrm{req}}$ *of the fuzzy safety set.*

For example, $\mathrm{SF}_{\mathrm{req}} = 2$ corresponds to $\alpha = 0.5$; $\mathrm{SF}_{\mathrm{req}} = 4$ corresponds to $\alpha = 0.75$. Classical deterministic design therefore operates on a specific α-cut of the underlying fuzzy safety model, even when the analysis is performed with purely deterministic arithmetic.

4 Global Safety: Fuzzy Intersection and Worst-Case Analysis

4.1 Global safety margin

For a structure with m members the *global safety margin* is the degree to which *every* member is simultaneously safe:

$$\mathrm{SM}(P_1, P_2) \;=\; \min\{\mathrm{SM}_1(P_1, P_2),\; \ldots,\; \mathrm{SM}_m(P_1, P_2)\}. \tag{2}$$

The minimum is taken because the structure fails as soon as any single member fails — a series system in reliability language [14].

4.2 Equivalence with fuzzy intersection

Fuzzy Intersection [,]

The standard fuzzy intersection (minimum t-norm) of fuzzy sets $\tilde{A}$ and $\tilde{B}$ on X is

$$\mu_{\tilde{A} \cap \tilde{B}}(x) = \min\{\mu_{\tilde{A}}(x),\; \mu_{\tilde{B}}(x)\}.$$

Global Worst-Case Analysis = Fuzzy Intersection

The global safety membership function is the fuzzy intersection of all member-wise safety sets:

$$\mu_{\text{safe}}(P_1, P_2) \;=\; \text{SM}(P_1, P_2) \;=\; \bigcap_{i=1}^{m} \tilde{S}_i \;:=\; \min_{i=1,\ldots,m} \text{SM}_i(P_1, P_2).$$

Classical worst-case structural analysis — finding the load combination that drives the most critical member to its allowable stress — is therefore a direct application of the fuzzy-intersection formula (2).

The minimum aggregator is the standard idempotent t-norm (Gödel t-norm) and recovers classical set intersection on crisp sets [6]. More general t-norms (product, Łukasiewicz) produce alternative safety models, each with its own engineering interpretation.

5 Five-Bar Plane Truss: Detailed Example

5.1 Truss geometry and equilibrium

Consider a plane truss with five bars subjected to two independent loads: a horizontal load P_1 at node A and a vertical (downward) load P_2 at node B. The geometry is shown in Figure 1.

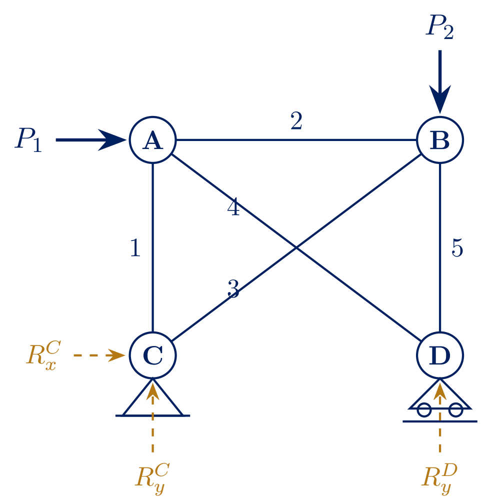

Figure 1: Five-bar plane truss. Nodes: A (top-left), B (top-right), C (bottom-left, pin support), D (bottom-right, roller support). Bar numbering: $1 = AC$, $2 = AB$, $3 = CB$, $4 = AD$, $5 = BD$.

From the equilibrium equations (derived by the method of joints [17]) the internal forces are

$$N_1 = 0, \tag{3}$$
$$N_2 = -P_1, \tag{4}$$
$$N_3 = \sqrt{2}\,P_1, \tag{5}$$
$$N_4 = 0, \tag{6}$$
$$N_5 = -(P_1 + P_2). \tag{7}$$

Members N_1 (bar AC, left vertical) and N_4 (bar AD, left diagonal) are *zero-force members* for this loading configuration: the horizontal pin at D has no horizontal reaction, which forces $N_4 = 0$ from joint-D x-equilibrium, and $N_1 = 0$ then follows from joint-A y-equilibrium.

The three structurally active members are therefore:

- bar 2 (AB, top horizontal member), carrying $N_2 = -P_1$;
- bar 3 (CB, diagonal), carrying $N_3 = \sqrt{2}\,P_1$;
- bar 5 (BD, right vertical member), carrying $N_5 = -(P_1 + P_2)$.

5.2 Member safety margins

Assuming a common cross-sectional area A and allowable stress $\sigma_{\max}$, the safety margins of the three active bars are

$$\mathrm{SM}_2(P_1, P_2) = \max\Big(0, 1 - \frac{|P_1|}{A\sigma_{\max}}\Big), \quad (8)$$

$$\mathrm{SM}_3(P_1, P_2) = \max\Big(0, 1 - \frac{\sqrt{2}|P_1|}{A\sigma_{\max}}\Big), \quad (9)$$

$$\mathrm{SM}_5(P_1, P_2) = \max\Big(0, 1 - \frac{|P_1 + P_2|}{A\sigma_{\max}}\Big). \quad (10)$$

Remark

SM_2 and SM_3 are functions of P_1 alone; their graphs over the (P_1, P_2) plane are ridge surfaces, constant in the P_2 direction. SM_5 depends on the diagonal combination $P_1 + P_2$, producing a tent-shaped surface symmetric about the line $P_1 + P_2 = 0$.

5.3 2D membership functions as functions of P_1

Figures 2–3 show the safety margins SM_2 and SM_3 as functions of P_1 alone, together with the α-cut threshold ($\alpha = 0.5$, $A\sigma_{\max} = 10$). The support of SM_2 is $|P_1| \le 10$; the support of SM_3 is the narrower interval $|P_1| \le 10/\sqrt{2} \approx 7.07$.

5.4 Ridge surfaces: $\mathrm{SM}_2(P_1, P_2)$ and $\mathrm{SM}_3(P_1, P_2)$

Because SM_2 and SM_3 depend only on P_1, their graphs over the (P_1, P_2) plane are ridge surfaces, infinite in the P_2 direction (Figures 4–5).

5.5 Tent surface: $\mathrm{SM}_5(P_1, P_2)$

Bar 5 couples both loads through $N_5 = -(P_1 + P_2)$, giving a tent-shaped surface symmetric about the diagonal $P_1 + P_2 = 0$ (Figure 6).

5.6 Global safety surface $\mathrm{SM}(P_1, P_2)$

The global safety surface is the pointwise minimum of all three member surfaces (Figure 7). The characteristic shape — two sharp ridges meeting at the origin — reflects the fact that bar 3 (the diagonal) is the binding member for loads along the P_1 axis, while bar 5 becomes binding when $P_1 + P_2$ is large.

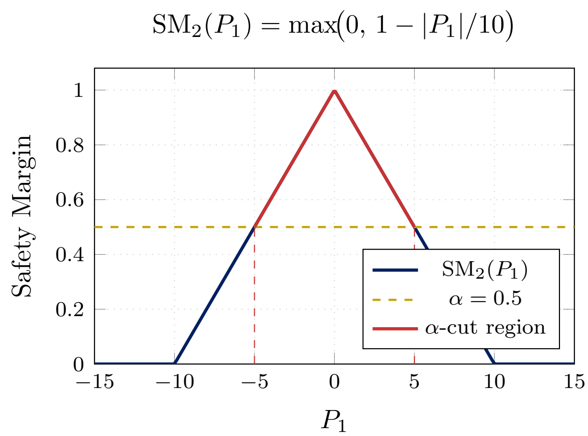

Figure 2: 1D membership function $\mathrm{SM}_2(P_1)$ for bar 2 (horizontal member AB). The α-cut at $\alpha = 0.5$ gives $|P_1| \leq 5 = (1-\alpha)A\sigma_{\max}$.

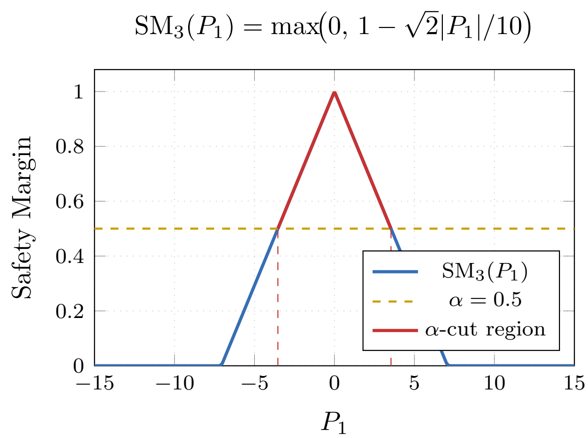

Figure 3: 1D membership function $\mathrm{SM}_3(P_1)$ for bar 3 (diagonal CB). The α-cut at $\alpha = 0.5$ gives $|P_1| \leq 5/\sqrt{2} \approx 3.54$, a narrower interval than bar 2 because bar 3 carries a $\sqrt{2}$-amplified force.

5.7 Global safety margin and its alpha-cut

The global safety margin is

$$\mathrm{SM}(P_1, P_2) = \min\{\mathrm{SM}_2(P_1, P_2),\ \mathrm{SM}_3(P_1, P_2),\ \mathrm{SM}_5(P_1, P_2)\}. \tag{11}$$

Applying Theorem 3.3, the α-cut $[\tilde{S}]_\alpha$ is defined by the simultaneous inequalities:

$$|P_1| \leq (1-\alpha)\, A\, \sigma_{\max}, \tag{12}$$
$$\sqrt{2}|P_1| \leq (1-\alpha)\, A\, \sigma_{\max}, \tag{13}$$
$$|P_1 + P_2| \leq (1-\alpha)\, A\, \sigma_{\max}. \tag{14}$$

Inequality (13) (from the diagonal bar 3) is more restrictive than (12) (from bar 2), so the binding vertical-strip constraint comes from bar 3. The diagonal strip from bar 5 intersects this vertical strip, producing a hexagonal feasible region in the (P_1, P_2) plane.

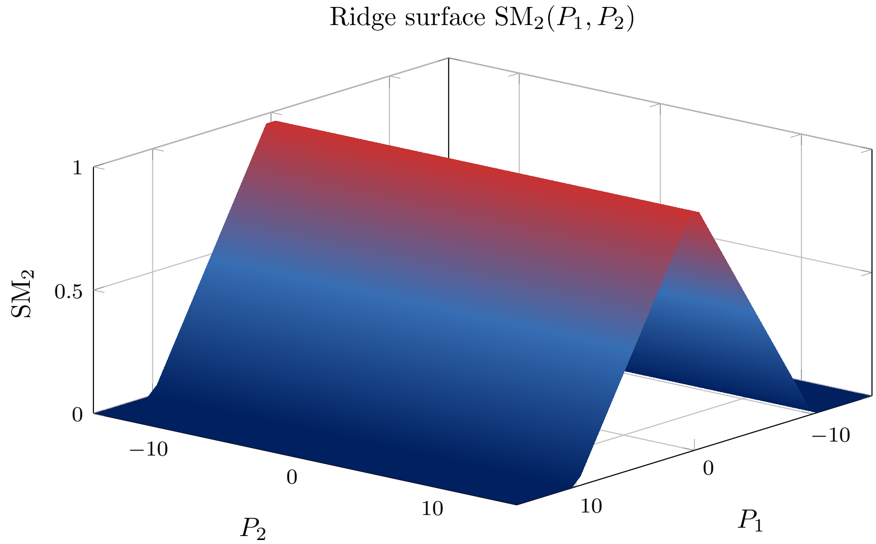

Figure 4: Ridge surface $\mathrm{SM}_2(P_1, P_2)$: the safety margin of bar 2 is independent of P_2, producing a tent shape in the P_1 direction that is constant (extruded) in the P_2 direction.

5.8 Schematic of individual and global alpha-cuts

The shaded hexagon is the α-cut of the global fuzzy safety set: the set of load combinations (P_1, P_2) for which *every* bar satisfies $\mathrm{SM}_i \geq \alpha$.

6 The Extension Principle in Structural Analysis

6.1 Zadeh's extension principle

Extension Principle [,]

Let $f : X \to Y$ and let $\tilde{A}$ be a fuzzy set on X with membership function $\mu_{\tilde{A}}$. The *image* of $\tilde{A}$ under f is the fuzzy set $\tilde{B} = f(\tilde{A})$ on Y with membership function

$$\mu_{\tilde{B}}(y) = \sup_{\substack{x \in X \\ f(x)=y}} \mu_{\tilde{A}}(x).$$

For a vector function $f : X_1 \times \cdots \times X_n \to Y$ and independent fuzzy inputs $\tilde{X}_1, \ldots, \tilde{X}_n$:

$$\mu_{\tilde{B}}(y) = \sup_{\substack{(x_1,\ldots,x_n) \\ f(x_1,\ldots,x_n)=y}} \min\bigl\{\mu_{\tilde{X}_1}(x_1), \ldots, \mu_{\tilde{X}_n}(x_n)\bigr\}.$$

6.2 Load-to-force propagation as the extension principle

Suppose the loads P_1, P_2 are themselves fuzzy, characterised by membership functions μ_{P_1} and μ_{P_2} (representing, for example, uncertain or variable loading). The equilibrium equations $N_i = N_i(P_1, P_2)$ then define a functional relationship between the fuzzy loads and the fuzzy internal forces.

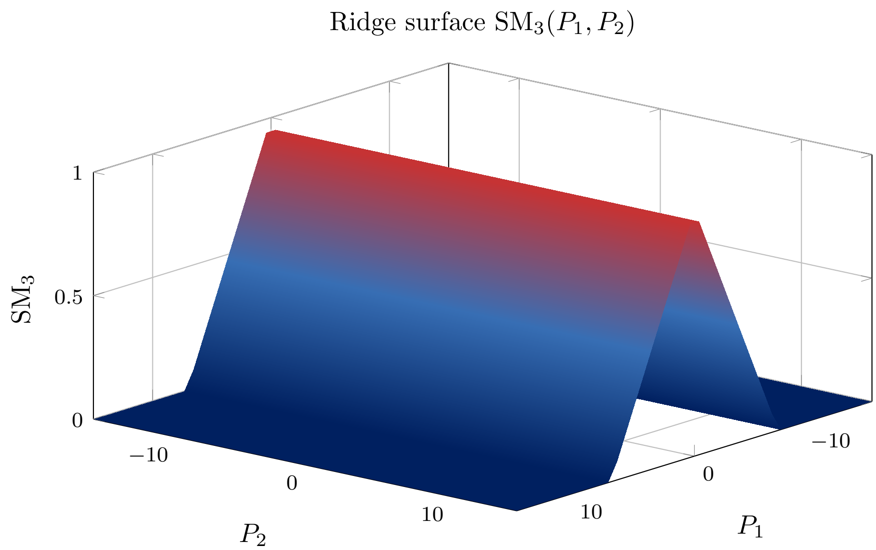

Figure 5: Ridge surface $\mathrm{SM}_3(P_1, P_2)$: narrower ridge than SM_2 because bar 3 (diagonal) carries a $\sqrt{2}$-amplified version of P_1.

Applying the extension principle gives the membership function of the internal force in bar i:

$$\mu_{N_i}(n) = \sup_{\substack{(P_1,P_2) \\ N_i(P_1,P_2)=n}} \min\{\mu_{P_1}(P_1),\ \mu_{P_2}(P_2)\}.$$

6.3 Safety margin of bar 5 expressed via its axial force

Consider bar 5 with equilibrium equation (7):

$$N_5 = -(P_1 + P_2).$$

For fixed $N_5 = n_5$, this constraint restricts the load pair to the line $P_1 + P_2 = -n_5$. In terms of N_5 alone, the safety margin can be written as

$$\mathrm{SM}_5^*(N_5) \ = \ \max\Big(0,\ 1 - \frac{|N_5|}{A\,\sigma_{\max}}\Big). \tag{15}$$

Equation (15) is consistent with the extension-principle viewpoint: the safety measure on the load space is transported through the equilibrium map $N_5(P_1, P_2)$ to a safety measure on the force space. For each target value n_5, the supremum in the extension formula ranges over all load pairs satisfying $P_1 + P_2 = -n_5$, and the resulting safety value is $\mathrm{SM}_5^*(n_5)$ as given by (15).

7 The Dependency Problem and Global Consistency Constraints

7.1 Independence assumption and its failure

Given the individual representations

$$\mathrm{SM}_i^*(N_i) \ = \ \max\Big(0,\ 1 - \frac{|N_i|}{A_i\,\sigma_{\max}}\Big),$$

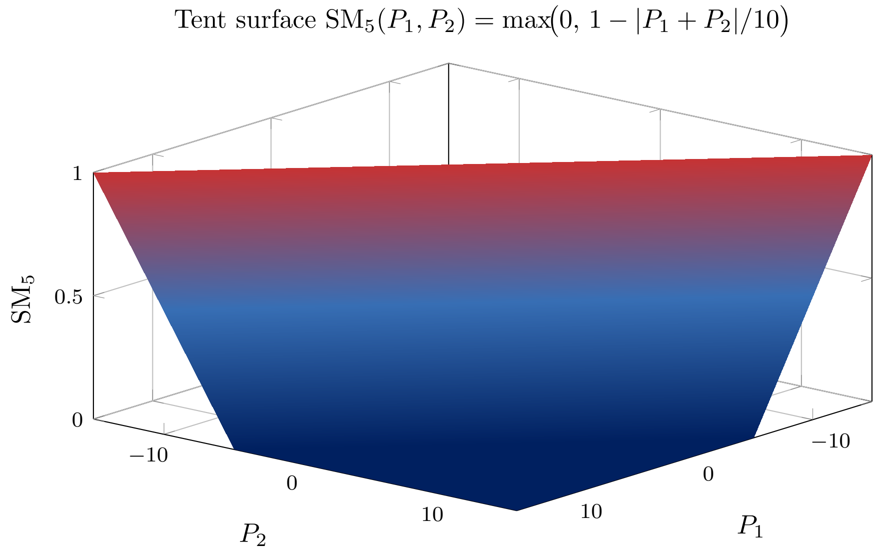

Figure 6: Tent surface $\mathrm{SM}_5(P_1, P_2)$: the safety margin of bar 5 is constant along lines $P_1+P_2 = \text{const}$, producing a ridge parallel to the anti-diagonal direction. The ridge peak lies along the line $P_1+P_2 = 0$.

one might attempt to form the global safety margin as

$$\mathrm{SM}^*(N_1, N_2, N_3, N_4, N_5) = \min\big\{\mathrm{SM}_1^*(N_1), \ldots, \mathrm{SM}_5^*(N_5)\big\}. \tag{16}$$

However, the axial forces N_i are *not independent*: they all depend on the same loads (P_1, P_2) through the equilibrium equations (4)–(7). Formula (16) treats them as though they were independent variables, which leads to an overly conservative (enlarged) solution set.

7.2 The dependency problem in interval arithmetic

This phenomenon is well known in interval arithmetic as the *dependency problem* [11, 12, 13]: when the same underlying variable appears in multiple terms of an expression, naive interval evaluation treats each occurrence as independent, introducing artificial overestimation.

A canonical example: if $x \in [0, 1]$, then $x - x = 0$, but interval arithmetic yields $[0, 1] - [0, 1] = [-1, 1]$. The same inflation occurs when P_1 appears in both $N_4 = \sqrt{2}P_1$ and $N_5 = -(P_1 + P_2)$ and these forces are subsequently combined without tracking the common source.

7.3 Physical interpretation

From a structural mechanics standpoint the dependency problem has a clear meaning: the forces in different members are linked through global equilibrium, not arbitrary combinations. Treating them as independent variables admits physically impossible load states (e.g. N_5 large while N_4 is small, violating equilibrium).

Correct Global Safety via Consistency Constraint

The correct global safety margin is

$$\mathrm{SM}(P_1, P_2) = \min_i \mathrm{SM}_i^*\big(N_i(P_1, P_2)\big),$$

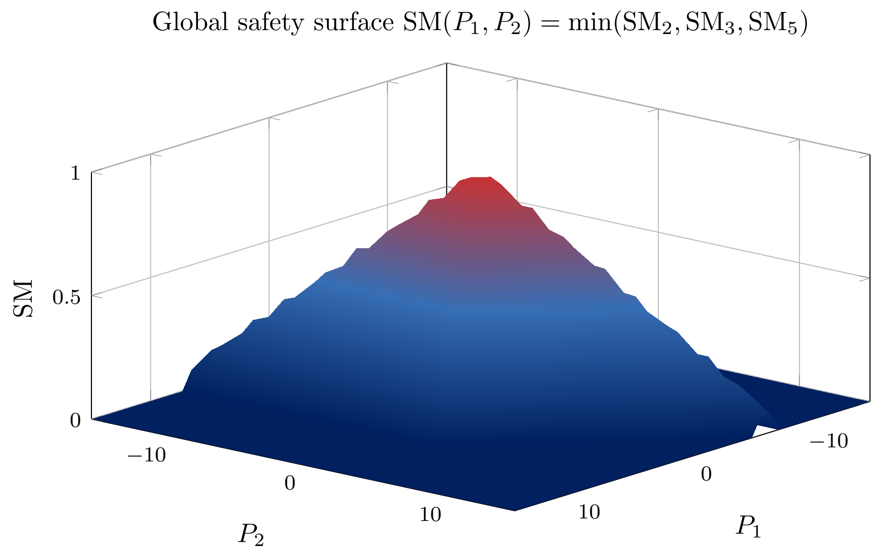

Figure 7: Global safety surface $\mathrm{SM}(P_1, P_2) = \min\{\mathrm{SM}_2, \mathrm{SM}_3, \mathrm{SM}_5\}$. The surface is bounded by the diagonal bar 3 along the P_1 axis and by bar 5 along the anti-diagonal direction. A horizontal slice at height α produces the hexagonal α-cut region shown in Figure 8.

where the minimisation is subject to the *equilibrium constraints* $N_i = N_i(P_1, P_2)$. For the truss of Section 5, the active members give:

$$\mathrm{SM}(P_1, P_2) = \min\big\{\mathrm{SM}_2^*(N_2),\ \mathrm{SM}_3^*(N_3),\ \mathrm{SM}_5^*(N_5)\big\} \quad \text{with} \quad \begin{cases} N_2 = -P_1, \\ N_3 = \sqrt{2}\, P_1, \\ N_5 = -(P_1 + P_2). \end{cases}$$

Members N_1 and N_4 are zero-force members and contribute $\mathrm{SM}_1^* = \mathrm{SM}_4^* = 1$ at all load levels, so they do not affect the global minimum. The equilibrium constraints play the role of the "compatibility conditions" that resolve the dependency problem in fuzzy arithmetic.

Theorem 7.3 shows why direct fuzzy or interval computations applied naively to individual bar formulas may give conservative results: the cross-member dependencies encoded in the structural equilibrium must be preserved explicitly.

8 Safety Margin as a Multivalued Degree of Truth

8.1 Many-valued logic

Classical propositional logic assigns to every proposition a truth value in $\{0, 1\}$.

Many-Valued Conjunction [,]

In infinite-valued many-valued logics the truth values form the unit interval $[0, 1]$. The *lattice-theoretic* (Gödel/Zadeh) conjunction is

$$p \wedge q = \min(p, q).$$

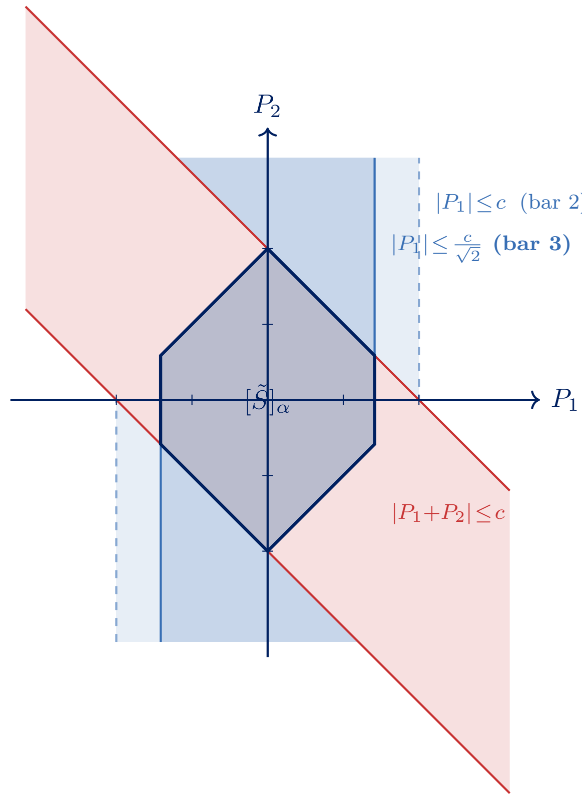

Figure 8: α-cuts of the three active member safety sets overlaid in the (P_1, P_2) load plane ($\alpha = 0.5$, $A\sigma_{\max} = 1$). Light blue: bar-2 strip $|P_1| \le c$. Medium blue: bar-3 strip $|P_1| \le c/\sqrt{2}$ (binding). Pink: bar-5 diagonal strip $|P_1 + P_2| \le c$. Navy hexagon: global α-cut $[\tilde{S}]_\alpha$.

This is idempotent and coincides with classical conjunction on $\{0, 1\}$. It should be distinguished from the strong Łukasiewicz t-norm,

$$p \otimes q = \max(0, p + q - 1),$$

which is not idempotent but satisfies different algebraic laws [6, 9].

8.2 The safety margin as a degree of truth

Consider the proposition

$$\mathcal{P}(P_1, P_2): \quad \text{“The structure is safe under loads } P_1, P_2\text{.”}$$

In classical (Boolean) logic this proposition is either true or false. In the present framework it admits a graded truth value:

Interpretable Membership via Safety Margin

Define the degree of truth of $\mathcal{P}$ by

$$\mu(P_1, P_2) \; = \; \mathrm{SM}(P_1, P_2) \; = \; \min_{i=1,\dots,m} \mathrm{SM}_i(P_1, P_2). \tag{17}$$

Then:

1. $\mu : \mathbb{R}^2 \to [0, 1]$, as required by the unit-interval truth scale.
2. $\mu(P_1, P_2) = 1$ iff all bars are unloaded (vacuously safe).
3. $\mu(P_1, P_2) = 0$ iff at least one bar reaches its allowable stress.
4. $0 < \mu(P_1, P_2) < 1$ represents partial safety — the gradual transition from safe to unsafe states.
5. The minimum operator in (17) is the standard idempotent (Gödel/Zadeh) fuzzy conjunction, which coincides with classical Boolean conjunction on $\{0, 1\}$ and serves as the lattice meet in many-valued semantics [10]. It is distinct from the strong Łukasiewicz t-norm $\max(0, p + q - 1)$, which is non-idempotent.
6. The α-cut $[\tilde{S}]_\alpha$ is the set of arguments for which the proposition is "true to degree at least α," which corresponds precisely to the allowable-force region for safety factor $\mathrm{SF}_{\mathrm{req}} = 1/(1 - \alpha)$.

Physical Interpretability

Formula (17) provides a *fully interpretable* definition of the degree of membership in a fuzzy set. Every parameter has a direct physical meaning: A_i is a cross-section, $\sigma_{\max}$ is a material constant, $N_i(P_1, P_2)$ is an equilibrium force, and SM_i is the normalised stress reserve.
This contrasts with typical fuzzy models, where the shape of the membership function is chosen heuristically. Here the membership function is *derived* from the physics of the structure.

8.3 Summary of equivalences

The following table collects the mathematical identifications established in this report.

Structural Mechanics	**Fuzzy Algebra**	**Many-Valued Logic**
Safety margin $\mathrm{SM}_i(P) \in [0, 1]$	Membership function $\mu_i(P)$	Degree of truth of "bar i safe"
$\mathrm{SM}_i(P) \geq \alpha$	α-cut $[\tilde{S}_i]_\alpha$	Proposition is true to degree $\geq \alpha$
Safety factor $\mathrm{SF}_{\mathrm{req}}$	α-level $1 - 1/\mathrm{SF}_{\mathrm{req}}$	Acceptance threshold
Global worst-case minimum	Fuzzy intersection (Gödel min t-norm)	Lattice conjunction ($\wedge$)
Equilibrium map $N_i = N_i(P)$	Extension principle	Function extension to graded arg
Dependency / equilibrium constraint	Consistent argument in extension	Shared variable in compound for
Allowable force domain	Support / α-cut	Set of sufficiently true instances

9 Conclusion

We have shown, through a sequence of elementary mathematical steps, that classical structural safety analysis is a concrete realisation of fuzzy algebra and multivalued logic:

1. The **safety margin** $\mathrm{SM}_i(P_1, P_2)$ is a fuzzy membership function, mapping every load combination to a degree of safety in $[0, 1]$.
2. The **allowable-force condition** for a given safety factor is mathematically identical to an α-cut of the fuzzy safety set, with $\alpha = 1 - 1/\mathrm{SF}_{\mathrm{req}}$.
3. **Global worst-case analysis** — taking the minimum safety margin over all members — is the standard fuzzy intersection formula (Gödel/Zadeh conjunction, the idempotent lattice-meet t-norm).
4. **Load propagation through equilibrium** is an instance of Zadeh's extension principle.
5. The **dependency problem** in fuzzy/interval arithmetic corresponds to violating the equilibrium constraints; the correct global safety model enforces these constraints, recovering the exact classical result.
6. The function $\mu(P_1, P_2) = \mathrm{SM}(P_1, P_2)$ provides a *physically derived, fully interpretable* definition of the degree of membership — and hence of graded truth — without any arbitrary parameter choices.

This framework suggests that engineering safety analysis can serve as a natural test bed for studying fuzzy and many-valued logic: every formula has a direct physical meaning, numerical verification is straightforward, and the models can be learned from experimental data using standard regression methods.

Note on Related Conference Presentation

The results and ideas presented in this report were developed as part of a broader framework and presented at the **2026 Annual Conference of the North American Fuzzy Information Processing Society (NAFIPS 2026)**, held in El Paso, Texas, March 14–16, 2026. The related conference paper is:

> Andrew Pownuk, *A General Constrained Descriptor-Based Framework for Interpretable Modeling: Unifying Fuzzy Descriptions, Classification, and Physics-Informed Neural Networks*, Proceedings of NAFIPS 2026, El Paso, Texas, March 14–16, 2026. https://sites.google.com/view/nafips26/

The truss safety-margin example discussed in Section 5 illustrates one concrete instance of the general constrained descriptor framework developed in that work: the safety margin functions $\mathrm{SM}_i(P_1, P_2)$ are the descriptors, the equilibrium constraints are the physics-informed constraints, and the minimum aggregation is the fuzzy intersection that produces the global interpretable model.

References

[1] A. Pownuk, *A General Constrained Descriptor-Based Framework for Interpretable Modeling: Unifying Fuzzy Descriptions, Classification, and Physics-Informed Neural Networks*, Proceedings of the 2026 Annual Conference of the North American Fuzzy Information Processing Society (NAFIPS 2026), El Paso, Texas, March 14–16, 2026. https://sites.google.com/view/nafips26/

[2] L. A. Zadeh, *Fuzzy sets*, Information and Control, 8(3):338–353, 1965.

[3] L. A. Zadeh, *The concept of a linguistic variable and its application to approximate reasoning*, Information Sciences, 8(3):199–249, 1975.

[4] D. Dubois and H. Prade, *Possibility Theory: An Approach to Computerized Processing of Uncertainty*, Plenum Press, New York, 1988.

[5] G. J. Klir and B. Yuan, *Fuzzy Sets and Fuzzy Logic: Theory and Applications*, Prentice Hall, Upper Saddle River, NJ, 1995.

[6] E. P. Klement, R. Mesiar, and E. Pap, *Triangular Norms*, Kluwer Academic Publishers, Dordrecht, 2000.

[7] J. Łukasiewicz, *On three-valued logic* (O logice trójwartościowej), Ruch Filozoficzny, 5:170–171, 1920.

[8] E. L. Post, *Introduction to a general theory of elementary propositions*, American Journal of Mathematics, 43(3):163–185, 1921.

[9] P. Hájek, *Metamathematics of Fuzzy Logic*, Kluwer Academic Publishers, Dordrecht, 1998.

[10] S. Gottwald, *A Treatise on Many-Valued Logics*, Research Studies Press, Baldock, UK, 2001.

[11] R. E. Moore, *Interval Analysis*, Prentice Hall, Englewood Cliffs, NJ, 1966.

[12] A. Neumaier, *Interval Methods for Systems of Equations*, Cambridge University Press, Cambridge, 1990.

[13] L. Jaulin, M. Kieffer, O. Didrit, and E. Walter, *Applied Interval Analysis*, Springer, London, 2001.

[14] R. E. Melchers, *Structural Reliability Analysis and Prediction*, 2nd ed., John Wiley & Sons, Chichester, 1999.

[15] I. Elishakoff, *Safety Factors and Reliability: Friends or Foes?*, Kluwer Academic Publishers, Dordrecht, 2004.

[16] J. M. Gere and B. J. Goodno, *Mechanics of Materials*, 7th ed., Cengage Learning, Mason, OH, 2009.

[17] R. C. Hibbeler, *Structural Analysis*, 9th ed., Pearson, Hoboken, NJ, 2014.