Stress Analysis of a Singly Reinforced Concrete Beam with Uncertain Structural Parameters

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Abstract. This paper presents the efforts by the authors to introduce interval uncertainty in the stress analysis of reinforced concrete flexural members. A singly reinforced concrete beam with interval values of steel reinforcement and corresponding Young's modulus and subjected to an interval bending moment is taken up for analysis. Using extension principle, the internal moment of resistance of the beam is expressed as a function of interval values of stresses in concrete and steel. The stress distribution model for the cross section of the beam given by IS 456-2000 (Indian standard code of practice for plain and reinforced concrete) is modified for this purpose. The internal moment of resistance is then equated to the external bending moment due to interval loads acting on the beam. The stresses in concrete and steel are obtained as interval values for various combinations of interval values of structural parameters. The interval stresses and strains in concrete and steel obtained using combinatorial solution; search-based algorithm and sensitivity analysis are found to be in excellent agreement.

Keywords: interval stresses; stress distribution; sensitivity analysis; search-based algorithm

1. Introduction

Analysis of rectangular beams of reinforced concrete is based on nonlinear and/or discontinuous stress-strain relationships and such analyses are difficult to perform. Provided the nature of loading, the beam dimensions, the materials used and the quantity of reinforcement are known, the theory of reinforced concrete permits the analysis of stresses, strains, deflections, crack spacing and width and also the collapse load. Further, the aim of analyzing the beam is to locate the neutral axis depth, find out the stresses in compression concrete and tensile reinforcement and also compute the moment of resistance. The aim of the designer of reinforced concrete beams is to predict the entire spectrum of behavior in mathematical terms, identify the parameters which influence this behavior, and obtain the cracking, deflection and collapse limit loads. There are usually innumerable answers to a design problem. Thus the design is followed by analysis and a final selection is obtained by a process of iteration. Thus the design process becomes clear only when the process of analysis is learnt thoroughly.

In the traditional (deterministic) methods of analysis, all the parameters of the system are taken to be precisely known. In practice, however, there is always some degree of uncertainty associated with the actual values for structural parameters. As a consequence of this, the structural system will always exhibit some degree of uncertainty. A reliable approach to handle uncertainty in a structural system is the use of interval algebra. In this approach, uncertainties in structural parameters will be introduced as interval values i.e., the values are known to lie between two limits, but the exact values are unknown. Thus, the problem is of determining conservative intervals for the structural response. Though interval arithmetic was introduced by Moore (Moore, 1966), the application of interval concepts to structural analysis is more recent. Modeling with intervals provides a link between design and analysis where uncertainty may be represented by bounded sets of parameters. Interval computation has become a significant computing tool with the software packages developed in the past decade. In the present work, a singly-reinforced concrete beam with interval area of steel reinforcement and corresponding interval Young's modulus and subjected to an interval moment is taken up for analysis. Interval algebra is used to establish the bounds for the stresses and strains in steel and concrete.

2. Literature Survey

In the literature there are several methods for solution of equations with interval parameters. In the year 1966, Moore (1966) discussed the problem of solution of system of linear interval equations. Neumaier (1990) discussed several methods of solution of linear interval equations in his book. Ben-Haim and Elishakoff (1990) introduced ellipsoid uncertainty. System of linear interval equation with dependent parameters and symmetric matrix was discussed by Jansson (1991). In their work Köylüoglu, Cakmak, Nielsen (1995) applied the concept of interval matrix to solution of FEM equations with uncertain parameters. Rao and Chen (1998) developed a new search-based algorithm to solve a system of linear interval equations to account for uncertainties in engineering problems. The algorithm performs search operations with an accelerated step size in order to locate the optimal setting of the hull of the solution.

McWilliam (2000) described several method of solution of interval equations. Akpan *et. al* (2001) used response surface method in order to approximate fuzzy solution. Vertex solution methodology that was based on α-cut representation was used for the fuzzy analysis. Muhanna and Mullen (2001) handled uncertainty in mechanics problems on using an interval-based approach. Muhanna's algorithm is modified by Rama Rao (2006) to study the cumulative effect of multiple uncertainties on the structural response. Neumaier and Pownuk (2007) explored properties of positive definite interval matrices. Their algorithm works even for very large uncertainty in parameters. Skalna, Rama Rao and Pownuk (2007) investigated the solution of systems of fuzzy equations in structural mechanics.

Several models were proposed to describe the stress distribution in the cross section of a concrete beam subjected to pure flexure. Initially, the parabolic model was proposed by Hognestad (1955) in 1951. This was followed by an exponential model proposed by Smith and Young (1955) and Desai and Krishnan model (1964). These models are applicable to concretes with strength below 40 MPa. The Indian standard code of practice for plain and reinforced concrete IS 456-2000 (2000) allows the assumption of any suitable relationship between the compressive stress distribution in concrete and the strain in concrete i.e. rectangle, trapezoid, parabola or any other shape which results in prediction of strength in substantial agreement with the results of test.

Rama Rao and Pownuk (2007) made the initial efforts to introduce uncertainty in the stress analysis of reinforced concrete flexural members. A singly reinforced concrete beam subjected to an interval load is taken up for analysis. Using extension principle, the internal moment of resistance of the beam is expressed as a function of interval values of stresses in concrete and steel. The stress distribution model for the cross section of the beam given by IS 456-2000 is modified for this purpose. The internal moment of resistance is then equated to the external bending moment due to interval loads acting on the beam. The stresses and strains in concrete and steel are obtained as interval values. The sensitivity of stresses in steel and concrete to corresponding variation of interval values of load about its mean values is explored.

A study of the effect of multiple uncertainties on the stress distribution across the cross section of a singly-reinforced concrete member is taken up by the authors in the present work. The stress distribution model suggested by the Indian code IS 456-2000 is followed in the present study (Figure 1). Post cracking behavior up to Limit State of Serviceability is considered in the present work (Purushottaman, 1986).

3. Stress analysis of a singly reinforced concrete section

3.1 STRESS DISTRIBUTION DUE TO A CRISP MOMENT

A singly reinforced concrete section shown in Figure 1 with is taken up for analysis of stresses and strains in concrete and steel. The beam has a width of b and an effective depth of d. The beam is subjected to a maximum external moment M. Strain-distribution is linear and ε_{cc} is the strain in concrete at the extreme compression fiber and ε_s is the strain in steel. Let *x* be the neutral axis depth from the extreme compression fiber. The aim of analyzing the beam is to locate this neutral axis depth, find out the stresses in compression concrete and the tensile reinforcement and also compute the moment of resistance.

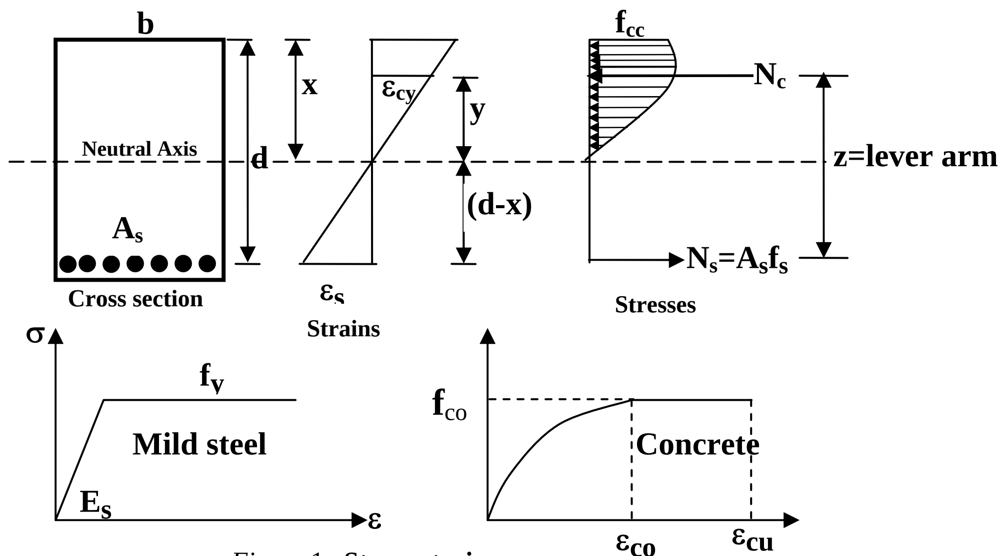

Figure 1. **Stress-strain curves**

The stress-distribution is concrete is parabolic and concrete in tension is neglected. The strain ε_{cy} at any level y below the neutral axis ($y \le x$) is

$$\varepsilon_{cy} = \left(\frac{y}{x}\right)\varepsilon_{cc} \tag{1}$$

The corresponding stress f_{cy} is

$$f_{cy} = f_{co}\left[2\left(\frac{\varepsilon_{cy}}{\varepsilon_{co}}\right) - \left(\frac{\varepsilon_{cy}}{\varepsilon_{co}}\right)^2\right] \quad \text{for} \quad \varepsilon_{cy} \le \varepsilon_{co} \ \text{and} \ f_{cy} = f_{co} \ \text{for} \ \varepsilon_{cy} = \varepsilon_{co} \tag{2}$$

Total compressive force in concrete N_c is given by

$$N_c = \int_{y=0}^{y=x} f_{cy} b dy = \left[C_1\varepsilon_{cc} - C_1\varepsilon_{cc}^2\right]x \tag{3}$$

where

$$C_1 = \left(\frac{bf_{co}}{\varepsilon_{co}}\right) \quad \text{and} \quad C_2 = \left(\frac{bf_{co}}{3\varepsilon_{co}^2}\right) \tag{4}$$

Tensile stress in steel

$$N_s = (A_s E_s)\varepsilon_{cc}\left(\frac{d-x}{x}\right) \tag{5}$$

If there are no external loads, the equation of longitudinal equilibrium, $N_s = N_c$ leads to the quadratic equation

$$\left[C_1 - C_2\varepsilon_{cc}\right]x^2 + A_s E_s x - A_s E_s d = 0 \tag{6}$$

Depth of resultant compressive force from the neutral axis $\overline{y}$ is given by

$$\overline{y} = \frac{\int_{y=0}^{y=x} bf_{cy}\,y\,dy}{\int_{y=0}^{y=x} bf_{cy}\,dy} = \frac{\left[\left(\frac{2C_1}{3}\right) - \left(\frac{3C_2}{4}\right)\varepsilon_{cc}\right]}{\left[C_1 - C_2\varepsilon_{cc}\right]}x \tag{7}$$

Internal resisting moment M_R is given by

$$M_R = N_c \times z = N_c \times (\overline{y} + d - x) \tag{8}$$

For equilibrium the external moment M is equated to the internal moment of resistance M_R as

$$M \le M_R \tag{9}$$

The neutral axis depth x can determined by solving equation (6) only when ε_{cc} is known. Thus a trial and error procedure is adopted where in ε_{cc} is assumed and the corresponding values of N_C , $\overline{y}$ and internal resisting moment M_R are obtained using equation (3), equation (7) and equation (8) such that equation (9) is satisfied.

Strain in steel $\varepsilon_s = \left(\frac{d-x}{x}\right)\varepsilon_{cc}$ (10)

Stress in steel $f_s = E_s \varepsilon_s = E_s \left(\dfrac{d-x}{x} \right) \varepsilon_{cc} \le 0.87 f_y$ (11)

Total tensile force in steel reinforcement = $N_s = A_s f_s$ (12)

4. Stress Distribution due to uncertain interval parameters

Consider the case of a singly reinforced concrete beam with interval values of area of steel reinforcement $\mathbf{A_s}$ with corresponding interval Young's modulus $\mathbf{E_s}$ and subjected to an interval external bending moment $\mathbf{M} = \left[\underline{M}, \overline{M} \right]$. The uncertainty in external moment arises out of uncertainty of loads acting on the beam. Correspondingly the resulting stresses and strains in concrete and steel are also uncertain and are modeled using interval numbers.

Using extension principle (Zadeh, 1965) all the equations developed in the previous section can be extended and made applicable to the interval case. The objective of the present study is to determine distribution of stresses and strain across the cross section of the beam. Two new approaches have been proposed for this purpose: a search based algorithm and a procedure based on Pownuk's sensitivity analysis (Pownuk, 2004).These methods are outlined as follows:

4.1 SEARCH-BASED ALGORITHM (SBA)

A search based algorithm (SBA) is developed to perform search operations with an accelerated step size in order to compute the optimal setting for the interval value of strain in concrete is $\boldsymbol{\varepsilon}_{cc} = \left[\underline{\varepsilon_{cc}}, \overline{\varepsilon}_{cc} \right]$. The algorithm is outlined below:

4.1.1 *Algorithm -1 (Search-based algorithm)*

a) The mid-value M of the given interval moment $\mathbf{M}$ is computed as $M = \dfrac{\underline{M} + \overline{M}}{2}$

b) The mid-value A_s of interval area of steel reinforcement $\mathbf{A_s}$ is computed as $A_s = \dfrac{\underline{A_s} + \overline{A_s}}{2}$

c) The mid-value E_s of interval Young's modulus of steel $\mathbf{E_s}$ is computed as $\mathrm{E_s} = \dfrac{\underline{\mathrm{E_s}} + \overline{\mathrm{E}}_\mathrm{s}}{2}$

d) Now the interval form of quadratic equation (6) given below is solved using the procedure outlined by Hansen and Walster (2002)

$$\left[C_1 - C_2\boldsymbol{\varepsilon}_{cc}\right]\mathbf{x}^2 + \mathbf{A_s E_s x} - \mathbf{A_s E_s} d = 0 \tag{13}$$

Various values of $\boldsymbol{\varepsilon}_{cc}$ are assumed and the neutral axis depth $\mathbf{x}$ and the corresponding values of $\mathbf{N_c}, \bar{\mathbf{y}}$ and $\mathbf{M_R}$ are determined by using a trial and error procedure outlined in the previous section.

e) The interval strain in concrete $\boldsymbol{\varepsilon}_{cc}$ is initially approximated as the point interval $\left[\varepsilon_{cc}, \varepsilon_{cc}\right]$.

f) The lower and upper bounds of $\boldsymbol{\varepsilon}_{cc}$ are obtained as $\boldsymbol{\varepsilon}_{cc} = \left[\varepsilon_{cc} - \lambda_1 d\underline{\varepsilon}, \varepsilon_{cc} + \lambda_2 d\bar{\varepsilon}\right]$ where $d\underline{\varepsilon}$ and $d\bar{\varepsilon}$ are the step sizes in strain to obtain the lower and upper bounds, λ_1 and λ_2 being the corresponding multipliers. Initially λ_1 and λ_2 are taken as unity.

g) While both λ_1, λ_2 are non-zero, $d\underline{\varepsilon}$ and $d\bar{\varepsilon}$ are incremented and $\boldsymbol{\varepsilon}_{cc}$ is computed. The procedure is continued iteratively till the interval form of (9) i.e. $\mathbf{M} \le \mathbf{M_R}$ is satisfied. The computations performed are outlined as follows:

   1) The interval values of $\mathbf{x}, \bar{\mathbf{y}}, \mathbf{z}, \mathbf{N_c}$ and the interval internal resisting moment $\mathbf{M_R} = [\underline{M}_R, \bar{M}_R]$ are computed. If η is a very small number

   2) λ_1 is set to zero if $\left|\dfrac{\underline{\underline{M}}_R - \underline{M}}{\underline{M}_R}\right| \le \eta$ (14)

   3) λ_2 is set to zero if $\left|\dfrac{\bar{M}_R - \bar{M}}{\bar{M}_R}\right| \le \eta$ (15)

   4) The search is discontinued when $\lambda_1 = \lambda_2 = 0$.

4.2 SENSITIVITY ANALYSIS METHOD

4.2.1 *Extreme values of ε_{cc} and x.*

Unknown variables ε_{cc} and x can be found from the system of equation (8) and equilibrium equation $N_s = N_c$. Let us introduce a new notation

$$\begin{cases} F_1 = F_1(\varepsilon_{cc}, x, p_1, ..., p_m) = M_R - N_c \cdot (\bar{y} + d - x) = 0 \\ F_2 = F_2(\varepsilon_{cc}, x, p_1, ..., p_m) = N_s - N_c = 0 \end{cases} \tag{16}$$

where $p_1 = M$, $p_2 = f_{co}$, $p_3 = A_S$, $p_4 = \varepsilon_{co}$, $p_5 = E_S$, $p_6 = b$, $p_7 = d$.

Because the problem is relatively simple and the intervals $\left[\underline{p}_i, \overline{p}_i \right]$ are usually narrow, then it is possible to solve the problem using sensitivity analysis method (Pownuk, 2004). Let us calculate sensitivity of the solution with respect to the parameter p_i .

$$\frac{\partial}{\partial p_i} F_1 = \frac{\partial F_1}{\partial \varepsilon_{cc}} \frac{\partial \varepsilon_{cc}}{\partial p_i} + \frac{\partial F_1}{\partial x} \frac{\partial x}{\partial p_i} + \frac{\partial F_1}{\partial p_i} = 0 \tag{17}$$

$$\frac{\partial}{\partial p_i} F_2 = \frac{\partial F_2}{\partial \varepsilon_{cc}} \frac{\partial \varepsilon_{cc}}{\partial p_i} + \frac{\partial F_2}{\partial x} \frac{\partial x}{\partial p_i} + \frac{\partial F_2}{\partial p_i} = 0 \tag{18}$$

In matrix form

$$\begin{bmatrix} \frac{\partial F_1}{\partial \varepsilon_{cc}} & \frac{\partial F_1}{\partial x} \\ \frac{\partial F_2}{\partial \varepsilon_{cc}} & \frac{\partial F_2}{\partial x} \end{bmatrix} \begin{bmatrix} \frac{\partial \varepsilon_{cc}}{\partial p_i} \\ \frac{\partial x}{\partial p_i} \end{bmatrix} = \begin{bmatrix} -\frac{\partial F_1}{\partial p_i} \\ -\frac{\partial F_2}{\partial p_i} \end{bmatrix} \tag{19}$$

Using Cramer's rule the solution is given by the following formulas

$$\frac{\partial \varepsilon_{cc}}{\partial p_i} = -\frac{\begin{vmatrix} \frac{\partial F_1}{\partial p_i} & \frac{\partial F_1}{\partial x} \\ \frac{\partial F_2}{\partial p_i} & \frac{\partial F_2}{\partial x} \end{vmatrix}}{\begin{vmatrix} \frac{\partial F_1}{\partial \varepsilon_{cc}} & \frac{\partial F_1}{\partial x} \\ \frac{\partial F_2}{\partial \varepsilon_{cc}} & \frac{\partial F_2}{\partial x} \end{vmatrix}} = -\frac{\frac{\partial (F_1, F_2)}{\partial (p_i, x)}}{\frac{\partial (F_1, F_2)}{\partial (\varepsilon_{cc}, x)}}, \quad \frac{\partial x}{\partial p_i} = -\frac{\begin{vmatrix} \frac{\partial F_1}{\partial \varepsilon_{cc}} & \frac{\partial F_1}{\partial p_i} \\ \frac{\partial F_2}{\partial \varepsilon_{cc}} & \frac{\partial F_2}{\partial p_i} \end{vmatrix}}{\begin{vmatrix} \frac{\partial F_1}{\partial \varepsilon_{cc}} & \frac{\partial F_1}{\partial x} \\ \frac{\partial F_2}{\partial \varepsilon_{cc}} & \frac{\partial F_2}{\partial x} \end{vmatrix}} = -\frac{\frac{\partial (F_1, F_2)}{\partial (\varepsilon_{cc}, p_i)}}{\frac{\partial (F_1, F_2)}{\partial (\varepsilon_{cc}, x)}} \tag{20}$$

If all Jacobians

$$\frac{\partial (F_1, F_2)}{\partial (\varepsilon_{cc}, x)}, \frac{\partial (F_1, F_2)}{\partial (p_i, x)}, \frac{\partial (F_1, F_2)}{\partial (\varepsilon_{cc}, p_i)} \tag{21}$$

are regular then the derivatives have constant sign and the relations $\varepsilon_{cc} = \varepsilon_{cc}(p_i)$, $x = x(p_i)$ are monotone. All variables p_i belong to known intervals $p_i \in \left[\underline{p}_i, \overline{p}_i\right]$ because of that sign of the Jacobians can be checked using interval global optimization method (Pownuk, 2004).

If

$$0 \neq \min_{\varepsilon_{cc} \in \left[\underline{\varepsilon}_{cc}, \overline{\varepsilon}_{cc}\right], x \in \left[\underline{x}, \overline{x}\right], p_1 \in \left[\underline{p}_1, \overline{p}_1\right], \ldots, p_m \in \left[\underline{p}_m, \overline{p}_m\right]} \left|\Delta(\varepsilon_{cc}, x, p_1, \ldots, p_m)\right| \tag{22}$$

then the Jacobian Δ is regular, where $\Delta = \frac{\partial(F_1, F_2)}{\partial(\varepsilon_{cc}, x)}$, $\Delta = \frac{\partial(F_1, F_2)}{\partial(p_i, x)}$ or $\Delta = \frac{\partial(F_1, F_2)}{\partial(\varepsilon_{cc}, p_i)}$. If the sign of the derivatives is constant then extreme values of the solution can be calculated using endpoints of the intervals $\left[\underline{p}_i, \overline{p}_i\right]$ and sensitivity analysis method (Pownuk, 2004). The whole algorithm of calculation is the following:

4.2.2 *Algorithm-2 (Sensitivity Analysis)*

1) Calculate mid point of the intervals $p_{i0} = \frac{\underline{p}_i + \overline{p}_i}{2}$.
2) Solve the system of equation (16) and calculate ε_{cc0}, x_0.
3) Calculate sensitivity of the solution $\frac{\partial \varepsilon_{cc}}{\partial p_i}$, $\frac{\partial x}{\partial p_i}$ from the system of equation (19).
4) If $\frac{\partial \varepsilon_{cc}}{\partial p_i} \geq 0$ then $p_i^{\min,\varepsilon_{cc}} = \underline{p}_i$, $p_i^{\max,\varepsilon_{cc}} = \overline{p}_i$, if $\frac{\partial \varepsilon_{cc}}{\partial p_i} < 0$ then $p_i^{\min,\varepsilon_{cc}} = \overline{p}_i$, $p_i^{\max,\varepsilon_{cc}} = \underline{p}_i$.
5) If $\frac{\partial x}{\partial p_i} \geq 0$ then $p_i^{\min,x} = \underline{p}_i$, $p_i^{\max,x} = \overline{p}_i$, if $\frac{\partial x}{\partial p_i} < 0$ then $p_i^{\min,x} = \overline{p}_i$, $p_i^{\max,x} = \underline{p}_i$.
6) Extreme values of ε_{cc}, x can be calculated as a solution of the following system of equations.
7) Verification of the results. If the derivatives have the same sign at the endpoints $p_i^{\min,x}, p_i^{\max,x}, p_i^{\min,\varepsilon_{cc}}, p_i^{\max,\varepsilon_{cc}}$ and in the midpoint then the solution is very reliable.

4.2.3 *Interval stress in extreme concrete fiber*

Sensitivity of stress in extreme concrete fiber f_{cc} can be calculated in the following way

$$\frac{\partial}{\partial p_i} f_{cc} = \frac{\partial f_{cc}}{\partial \varepsilon_{cc}} \frac{\partial \varepsilon_{cc}}{\partial p_i} + \frac{\partial f_{cc}}{\partial x} \frac{\partial x}{\partial p_i} + \frac{\partial f_{cc}}{\partial p_i} \quad (23)$$

where $\frac{\partial \varepsilon_{cc}}{\partial p_i}$ and $\frac{\partial x}{\partial p_i}$ are solution of the equation (19).

If $\frac{\partial f_{cc}}{\partial p_i} \geq 0$ then $p_i^{\min, f_{cc}} = \underline{p}_i$, $p_i^{\max, f_{cc}} = \overline{p}_i$, if $\frac{\partial f_{cc}}{\partial p_i} < 0$ then $p_i^{\min, f_{cc}} = \overline{p}_i$, $p_i^{\max, f_{cc}} = \underline{p}_i$.

$$\underline{f}_{cc} = f_{cc}\left(\varepsilon_{cc}^{\min, f_{cc}}, x^{\min, f_{cc}}, p_1^{\min, f_{cc}}, ..., p_m^{\min, f_{cc}}\right), \quad (24)$$

$$\overline{f}_{cc} = f_{cc}\left(\varepsilon_{cc}^{\max, f_{cc}}, x^{\max, f_{cc}}, p_1^{\max, f_{cc}}, ..., p_m^{\max, f_{cc}}\right) \quad (25)$$

In the midpoint sensitivity is equal to

$$\frac{\partial}{\partial M} f_{cc} = \frac{\partial f_{cc}}{\partial \varepsilon_{cc}} \frac{\partial \varepsilon_{cc}}{\partial M} + \frac{\partial f_{cc}}{\partial x} \frac{\partial x}{\partial M} + \frac{\partial f_{cc}}{\partial M} \quad (26)$$

Extreme values of stress in extreme concrete fiber calculated form the equations (24) and (25).

4.2.4 *Interval stress in steel*

Sensitivity of stress in steel f_s can be calculated in the following way

$$\frac{\partial}{\partial p_i} f_s = \frac{\partial f_s}{\partial \varepsilon_{cc}} \frac{\partial \varepsilon_{cc}}{\partial p_i} + \frac{\partial f_s}{\partial x} \frac{\partial x}{\partial p_i} + \frac{\partial f_s}{\partial p_i} \quad (27)$$

where $\frac{\partial \varepsilon_{cc}}{\partial p_i}$ and $\frac{\partial x}{\partial p_i}$ are solution of the equation (19).

If $\frac{\partial f_s}{\partial p_i} \geq 0$ then $p_i^{\min, f_s} = \underline{p}_i$, $p_i^{\max, f_s} = \overline{p}_i$, if $\frac{\partial f_s}{\partial p_i} < 0$ then $p_i^{\min, f_s} = \overline{p}_i$, $p_i^{\max, f_s} = \underline{p}_i$.

$$\underline{f}_s = f_s\left(\varepsilon_{cc}^{\min, f_s}, x^{\min, f_s}, p_1^{\min, f_s}, ..., p_m^{\min, f_s}\right), \quad (28)$$

$$\overline{f}_s = f_s\left(\varepsilon_{cc}^{\max,f_s}, x^{\max,f_s}, p_1^{\max,f_s},..., p_m^{\max,f_s}\right). \tag{29}$$

Sensitivity at the mid point is computed as

$$\frac{\partial}{\partial M} f_s = \frac{\partial f_s}{\partial \varepsilon_{cc}}\frac{\partial \varepsilon_{cc}}{\partial M} + \frac{\partial f_s}{\partial x}\frac{\partial x}{\partial M} + \frac{\partial f_s}{\partial M} \tag{30}$$

4.3 COMBINATORIAL SOLUTION

Combinatorial solution is obtained by considering the upper and lower bounds of the external interval moment and computing the corresponding deterministic values of ε_{cc}, x, $\overline{y}$, N_c and M_R are determined. The lower and upper values taken by these quantities are utilized to obtain the corresponding interval values of $\mathbf{x}$, $\overline{\mathbf{y}}$, $\mathbf{z}$, $\mathbf{N_c}$ and $\mathbf{M_R}$.

5. Example Problem

A singly reinforced beam with rectangular cross section is taken up to illustrate the validity of the above methods. The beam has the dimensions b = 300 mm and D = 550 mm and effective depth d = 500 mm. The beam is reinforced with 6 numbers of Tor50 bars of 25 mm diameter ($A_S = 6\times 491\ mm^2 = 2946 mm^2$).The bending moment acting on the beam is $\mathrm{M} = 100\mathrm{kNm}$. Allowable compressive stress in concrete f_{co} *is* $13.4\ N/mm^2$ and allowable strain in concrete ε_{co} is 0.002. Young’s modulus of steel E_S *is* $200GPa$.The stress-strain curves for concrete and steel as detailed in IS 456-2000 are adopted (Figure 1)

Interval uncertainty is considered in the bending moment, area of steel reinforcement and Young’s modulus of steel reinforcement. The corresponding membership functions are shown in Figure 2, Figure 3 and Figure 4 respectively. Interval values of bending moment, area of steel reinforcement and corresponding Young’s modulus can be extracted from these figures using α-cut approach at any desired level of uncertainty for use in the stress analysis. For example, corresponding to α = 0.8, the interval values considered are $\mathbf{M}$ = [98,102] kNm, $\mathbf{A_s}$ = [2917, 2975] mm^2 and $\mathrm{E_s} = [198, 202]\mathrm{GPa}$. The corresponding interval values of neutral axis depth, strain and stress in concrete and stress in steel reinforcement are computed at various levels of uncertainty and membership functions are plotted.

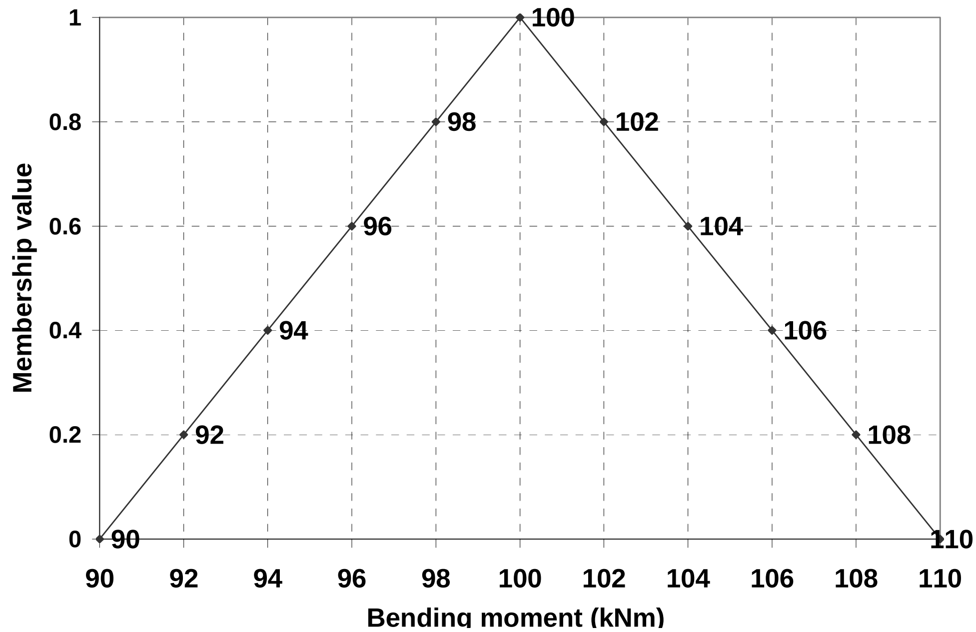

Figure 2. Membership function for bending moment

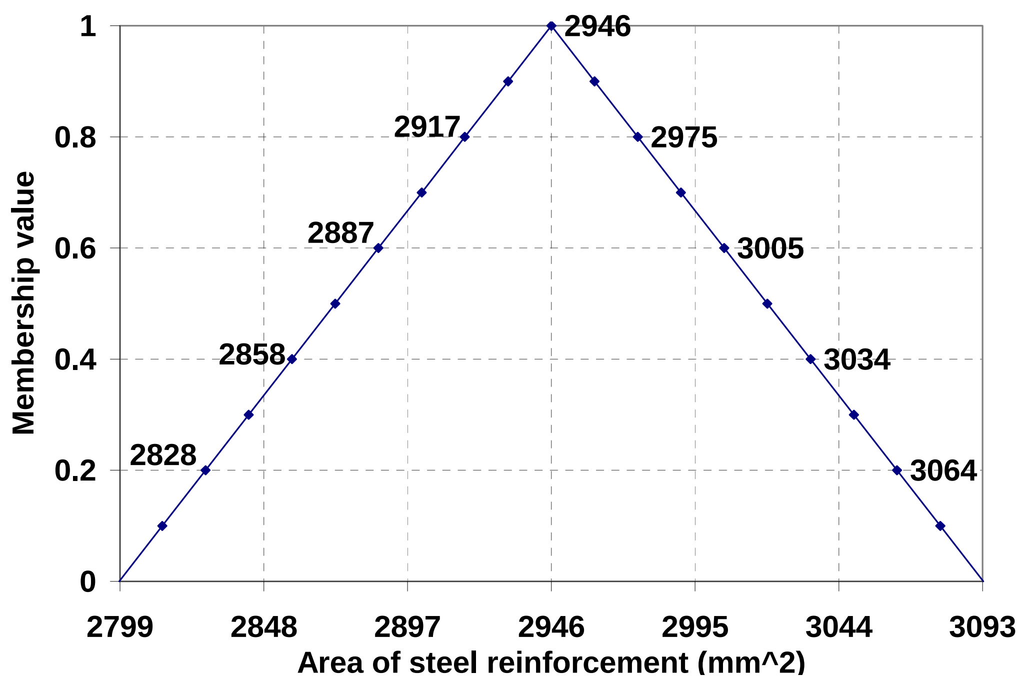

Figure 3. Membership function for area of steel reinforcement

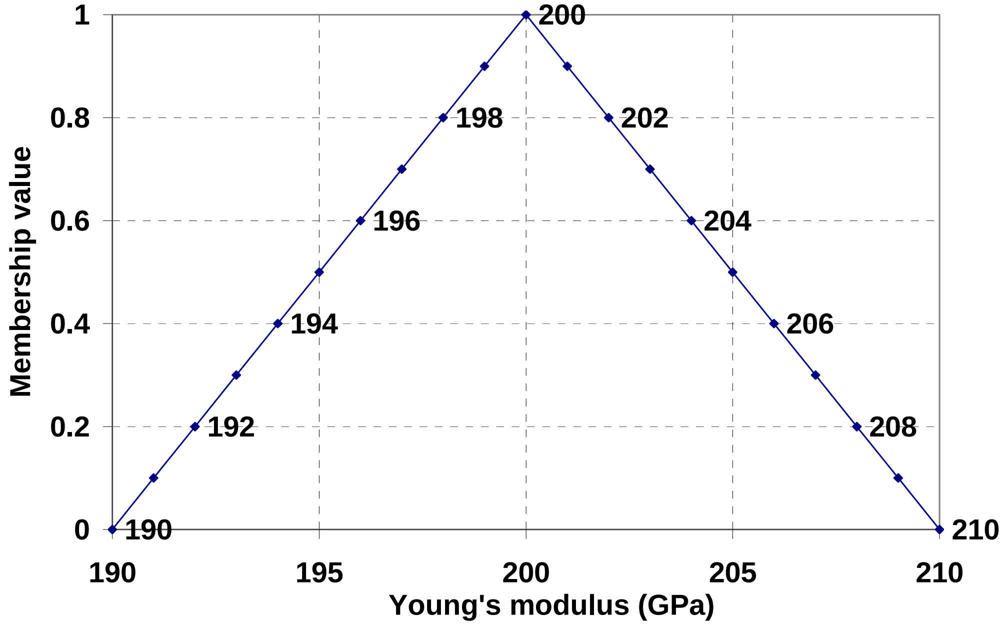

Figure 4. Membership function for Young's modulus of steel reinforcement

In the present study, interval values of neutral axis depth $\mathbf{x}$, strain $\boldsymbol{\varepsilon}_{cc}$ and stress $\mathbf{f}_{cc}$ in extreme compression fiber of concrete and stress in steel f_s computed using search-based algorithm (SBA) and sensitivity analysis (SA) approach for the following cases:

a) Case 1 :
External interval moment $\mathbf{M}$ = [96,104] kNm
Area of Steel reinforcement A_S = 2946 mm^2
Young's modulus of Steel reinforcement $E_S = 2\times10^5$ N/mm^2

b) Case 2:
External interval moment $\mathbf{M}$ = [90,110] kNm
Interval area of Steel reinforcement $\mathbf{A_s}$ = [0.9,1.1]*2946 mm^2
Young's modulus of Steel reinforcement $E_S = 2\times10^5$ N/mm^2

c) Case 3:
External interval moment $\mathbf{M}$ = [80,120] kNm
Area of Steel reinforcement A_S = 2946 mm^2
Interval Young's modulus of Steel reinforcement $\mathbf{E_s}$ = [0.98, 1.02]*2×10^5 N/mm^2

d) Case 4:
External interval moment $\mathbf{M}$ = [90,110] kNm
Interval area of Steel reinforcement $\mathbf{A}_s$ = [0.98, 1.02]*2946 mm^2
Interval Young's modulus of Steel reinforcement $\mathbf{E}_s$ = [0.98, 1.02]*2×10^5 N/mm^2

6. Results and Discussion

A web application is developed by the authors and is posted at the URL http://calculus.math.utep.edu/~andrzej/php/concrete-beam . Computations are performed using this web application. The screen capture of the web application is shown in Figure 5. The screen capture of the results obtained using this web application is shown in Figure 6.

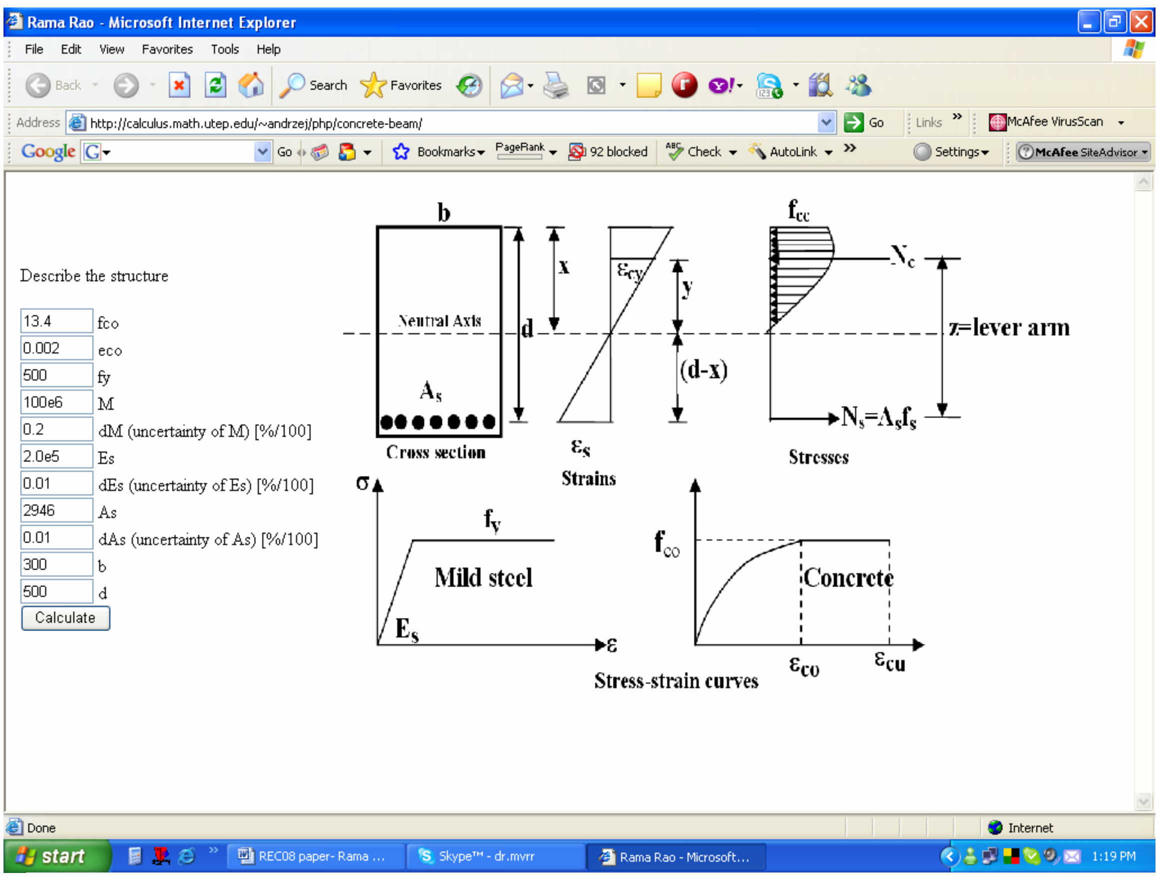

Figure 5. Web Application

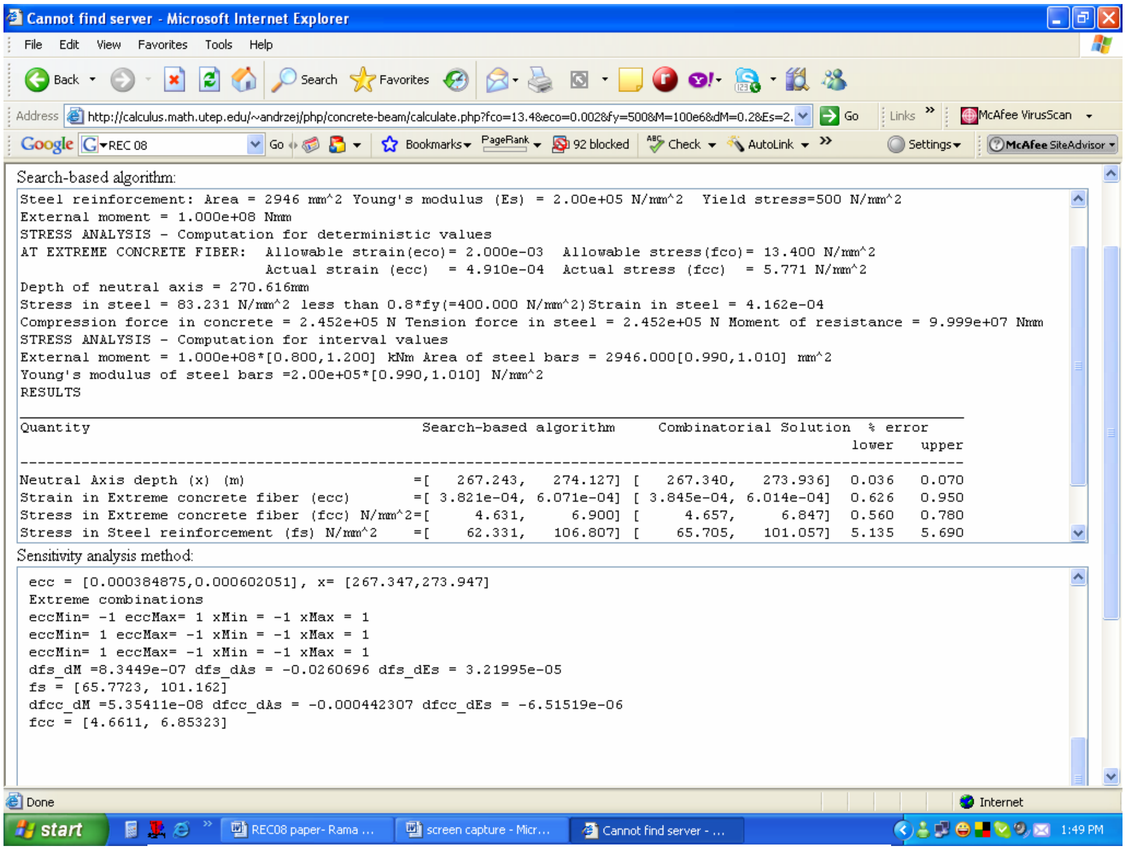

Figure 6. Results obtained using web application

Tables 1 *through* 4 present the results obtained using these three approaches viz. search-based algorithm, sensitivity analysis and combinatorial solution. Relative difference is computed for results obtained using search-based algorithm and sensitivity analysis in comparison with results obtained using combinatorial approach.

Table 1 presents the results obtained for Case 1. It is observed that the relative difference computed for search-based algorithm range from 0.003 percent to 0.416 percent while the corresponding relative difference computed using sensitivity analysis ranges from 0.0 percent to 0.005 percent. Table 2 presents the results obtained for Case 2. It is observed that the relative difference computed for search-based algorithm range from 0.179 percent to 2.2 percent while the corresponding relative difference computed using sensitivity analysis is almost zero. Table 3 presents the results obtained for Case 3. It is observed that the relative difference computed for search-based algorithm range from 0.039 percent to 7.567 percent while the corresponding relative difference computed using sensitivity analysis while the corresponding relative difference computed using sensitivity analysis is almost zero. Table 4 presents the results obtained for Case 4. It is observed that the relative difference computed for search-based algorithm range from 0.061 percent to 4.701 percent while the corresponding relative difference computed using sensitivity analysis is almost zero. Thus it is observed that the relative difference is very small. Thus these methods agree very well with the combinatorial solution.

Table. 1 Comparison of results obtained using the three approaches for $\mathbf{M}$ = [96,104]kNm

	$\varepsilon_{cc}\times10^{-4}$		f_{cc} (N/mm^2)		x (mm)		f_s (N/mm^2)	
	Lower	Upper	Lower	Upper	Lower	Upper	Lower	Upper
Combinatorial	4.699	5.123	5.557	5.985	270.291	270.945	79.870	86.612
Search based approach	4.704	5.117	5.562	5.980	270.299	270.937	79.543	86.972
% difference	0.106	0.117	0.090	0.084	0.003	0.003	0.409	0.416
Sensitivity Analysis	4.699	5.122	5.557	5.985	270.291	270.946	79.871	86.614
% difference	0.002	0.005	0.001	0.011	0.000	0.000	0.001	0.005

Table. 2 Comparison of results obtained using the three approaches for $\mathbf{M}$ = [90,110] kNm , A_s = [0.9,1.1]*2946mm^2

	$\varepsilon_{cc}\times10^{-4}$		f_{cc} (N/mm^2)		x (mm)		f_s (N/mm^2)	
	Lower	Upper	Lower	Upper	Lower	Upper	Lower	Upper
Combinatorial	4.279	5.601	5.121	6.454	261.07	279.374	68.467	101.149
Search based approach	4.261	5.631	5.102	6.483	260.693	279.874	67.029	103.374
% difference	0.421	0.536	0.371	0.449	0.144	0.179	2.100	2.200
Sensitivity Analysis	4.279	5.601	5.121	6.454	261.07	279.374	68.467	101.149
% difference	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

Table. 3 Comparison of results obtained using the three approaches for $M = [80,120]$ kNm and $E_s = [0.98,1.02]*2946mm^2$

	$\varepsilon_{cc} \times 10^{-4}$		f_{cc} (N/mm^2)		x (mm)		f_s (N/mm^2)	
	Lower	Upper	Lower	Upper	Lower	Upper	Lower	Upper
Combinatorial	3.848	6.021	4.661	6.853	267.338	273.939	66.339	100.286
Search based approach	3.821	6.071	4.631	6.901	267.235	274.119	61.707	107.875
% difference	0.702	0.830	0.644	0.700	0.039	0.066	6.982	7.567
Sensitivity Analysis	3.848	6.021	4.661	6.853	267.338	273.939	66.339	100.286
% difference	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

Table. 4 Comparison of results obtained using the three approaches for $M = [96,104]$ kNm, $A_s = [0.98,1.02]*2946mm^2$ $E_s = [0.98,1.02]*2.0e5kN/m$^2

	$\varepsilon_{cc} \times 10^{-4}$		f_{cc} (N/mm^2)		x (mm)		f_s (N/mm^2)	
	Lower	Upper	Lower	Upper	Lower	Upper	Lower	Upper
Combinatorial	4.651	5.178	5.507	6.040	266.935	274.239	78.303	88.379
Search based approach	4.643	5.188	5.499	6.051	266.771	274.423	74.797	92.534
% difference	0.172	0.193	0.145	0.182	0.061	0.067	4.477	4.701
Sensitivity Analysis	4.651	5.178	5.507	6.040	266.935	274.239	78.303	88.379
% difference	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

6.1 Combined membership function for stresses and Strains

Combined membership functions are plotted for stresses and strains in concrete and steel as well as neutral axis depth using the three approaches viz. search-based algorithm, sensitivity analysis and combinatorial approach. These membership functions are obtained using the procedure suggested by Moens and Vandepitte (2005). Figure 7 shows the plots of membership function for the depth of neutral axis. Combined membership functions for the strain and stress in extreme concrete fiber are presented in Figure 8 and Figure 9 respectively. Combined membership function for the stress in steel reinforcement is shown in Figure 10. It is observed that all these membership functions are triangular with linear variation of the response about the corresponding mean value. The plots of combined membership functions obtained using search-based approach and sensitivity analysis agree well with the membership functions plotted using combinatorial approach.

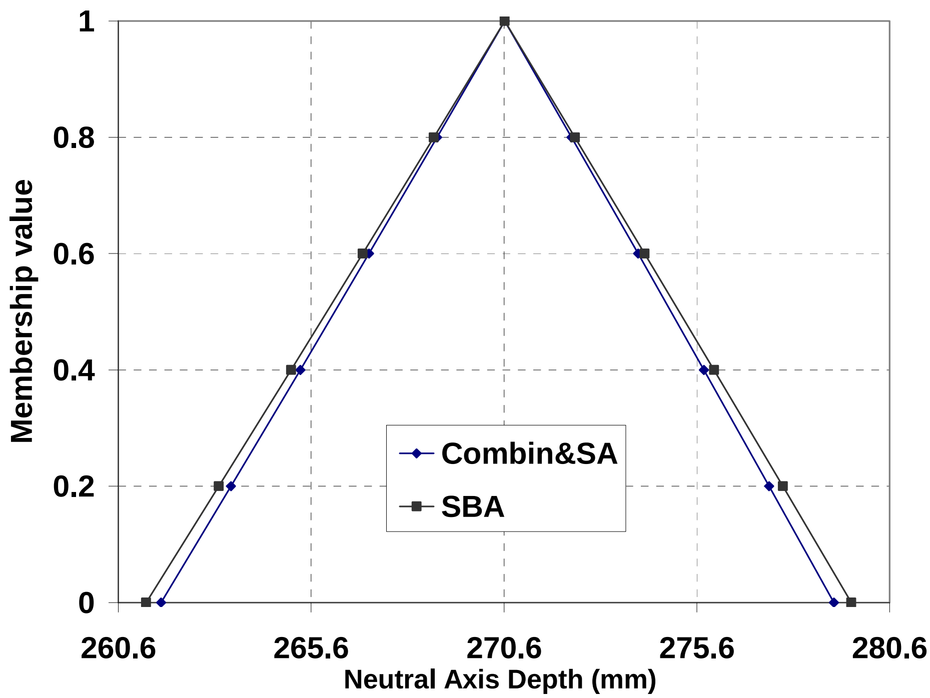

Figure .7 Combined membership function for neutral axis depth(x)

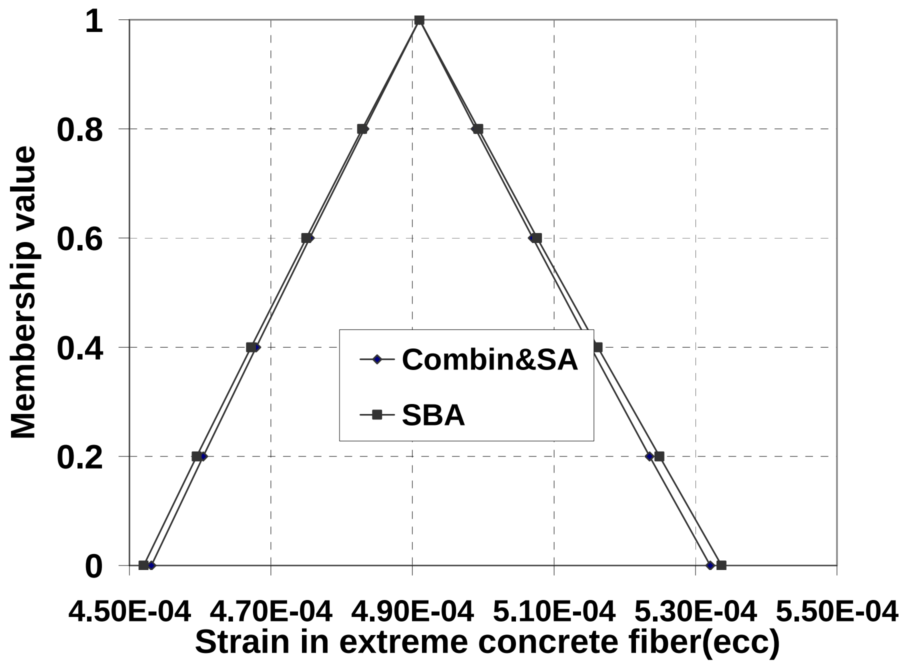

Figure .8 Combined membership function for strain (ε_{cc}) in extreme concrete fiber

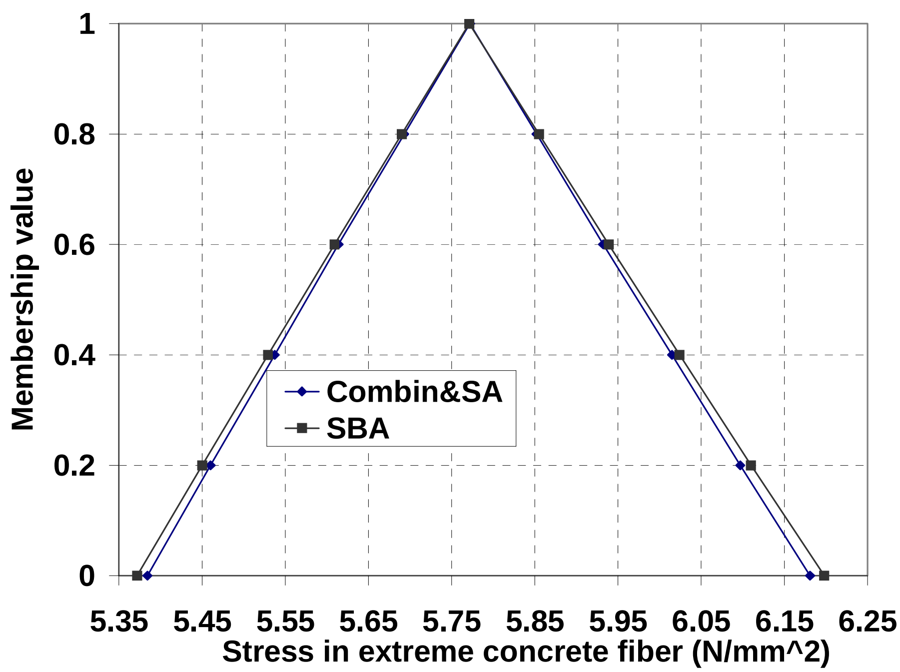

Figure 9. Combined membership function for stress (f_{cc}) in extreme concrete fiber

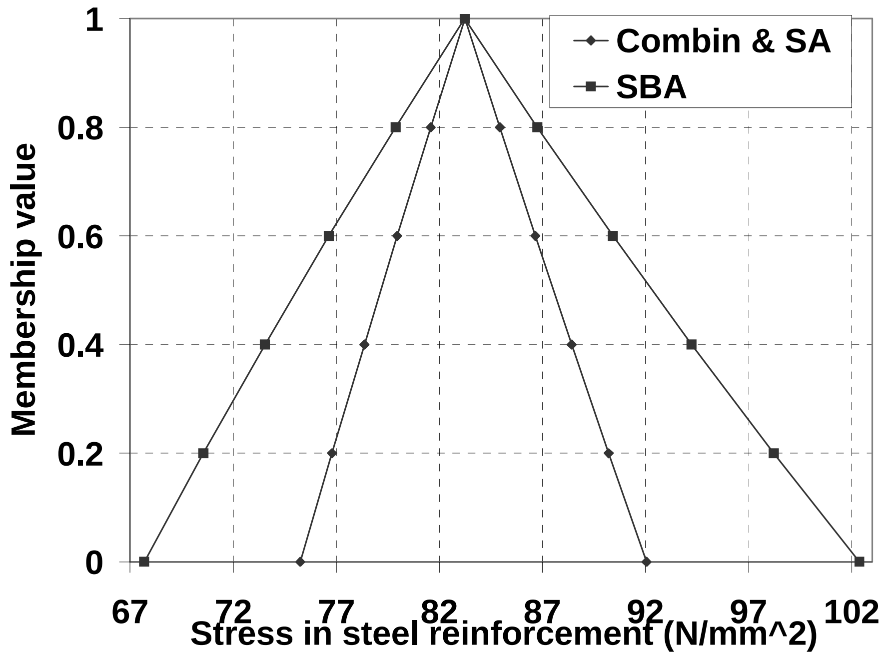

Figure .10 Combined membership function for stress (f_{cc}) in steel reinforcement

7. Conclusions

In the present paper, analysis of stresses in the cross section of a singly reinforced beam with interval values of area of steel reinforcement with corresponding interval Young's modulus and subjected to an interval external bending moment is taken up. The stress analysis is performed by three approaches viz. a search based algorithm and sensitivity analysis and combinatorial approach. It is observed that the results obtained are in excellent agreement. These approaches allow the designer to have a detailed knowledge about the effect of uncertainty on the stress distribution of the beam. The combined membership functions are plotted for neutral axis depth and stresses in concrete and steel and are found to be triangular.

Interval stresses and strains are also calculated using sensitivity analysis. The sign of the derivatives at the mid point and also at the endpoints is found to be same thus establishing the validity of the solution. More accurate monotonicity tests based on second and higher order derivatives (Pownuk, 2004) can also be used to establish sharp bounds on interval solution. Results with guaranteed accuracy can also be calculated using interval global optimization (Hansen, 1992 and Neumaier, 1990). Initial efforts by the authors in this direction gave encouraging results. Extended version of this paper will be published on the web page of the Department of Mathematical Science at the University of Texas at El Paso (http://www.math.utep.edu/preprints/).

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