Solution of Algebraic Equations by Using Self-Adaptive Computational Methods and Machine Learning

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Outline

Self Adaptive Computational Methods

Product rule (input information)

$$(f * g)' = f' * g + f * g'$$

After calculations (new theorem created automatically)

$$((f * g) * h)' = (f * g)' * h + (f * g) * h'$$

$$((f * g) * h)' = (f' * g + f * g') * h + (f * g) * h'$$

$$(f * g * h)' = (f' * g) * h + (f * g') * h + (f * g) * h'$$

New theorem can be used in exactly the same way like the original theorem.

$$(f * g * h)' = f' * g * h + f * g' * h + f * g * h'$$

Automated Theorem Proving

Lagrange Theorem.

- Function $f(x)$ is continuous in the interval $[a, b]$
- Function $f(x)$ is differentiable in the interval (a, b)

then exists $c \in (a, b)$ such that $\frac{f(b)-f(a)}{b-a} = f'(c)$

- Function $f(x)g(x)$ is continuous in the interval $[a, b]$
- Function $f(x)g(x)$ is differentiable in the interval (a, b)

then exists $c \in (a, b)$ such that
$\frac{f(b)g(b)-f(a)h(a)}{b-a} = f'(c)g(c) + f(c)g'(c)$

Differentiation

Input information

- $(f(x)+g(x))' = f'(x)+g'(x)$
- $(f(x)-g(x))' = f'(x)-g'(x)$
- $(f(x)g(x))' = f'(x)g(x)+f(x)g'(x)$
- $(f(x)/g(x))' = \frac{f'(x)g(x)-f(x)g'(x)}{g^2(x)}$
- $(f(g(x)))' = f'(g(x))g'(x)$
- arithmetic operations

Differentiation

Automatically generate formulas out of:

- List of arithmetical operations
- List of elementary functions and constants
- Information about the numer of arithmetical operations and functions compositions

Differentiation (Latex source)

```
\begin{equation}\cos\left(x\right)+x+2+\sin\left(2\right)\end{equation}
\begin{equation}\cos\left(x\right)+x+2+\sin\left(x\right)\end{equation}
\begin{equation}\cos\left(x\right)+x+2+\sin\left(\sin\left(x\right)\right)\end{equation}
\begin{equation}\cos\left(x\right)+x+2+\sin\left(\cos\left(x\right)\right)\end{equation}
\begin{equation}\cos\left(x\right)+x+2+\cos\left(2\right)\end{equation}
\begin{equation}\cos\left(x\right)+x+2+\cos\left(x\right)\end{equation}
\begin{equation}\cos\left(x\right)+x+2+\cos\left(\sin\left(x\right)\right)\end{equation}
\begin{equation}\cos\left(x\right)+x+2+\cos\left(\cos\left(x\right)\right)\end{equation}
\begin{equation}\cos\left(x\right)+x+2+2+2\end{equation}
\begin{equation}\cos\left(x\right)+x+2+2+x\end{equation}
\begin{equation}\cos\left(x\right)+x+2+2+\sin\left(x\right)\end{equation}
\begin{equation}\cos\left(x\right)+x+2+2+\cos\left(x\right)\end{equation}
\begin{equation}\cos\left(x\right)+x+2+x+x\end{equation}
\begin{equation}\cos\left(x\right)+x+2+x+\sin\left(x\right)\end{equation}
\begin{equation}\cos\left(x\right)+x+2+x+\cos\left(x\right)\end{equation}
\begin{equation}\cos\left(x\right)+x+2+\sin\left(x\right)+\sin\left(x\right)\end{equation}
\begin{equation}\cos\left(x\right)+x+2+\sin\left(x\right)+\cos\left(x\right)\end{equation}
\begin{equation}\cos\left(x\right)+x+2+\cos\left(x\right)+\cos\left(x\right)\end{equation}
\begin{equation}\cos\left(x\right)+x+x+x\end{equation}
```

Differentiation (step 1)

$$\cos(x) + (\cos(x) + \cos(2)) \tag{765}$$
$$\cos(x) + (\cos(x) + \cos(x)) \tag{766}$$
$$\cos(x) + (\cos(x) + \cos(\sin(x))) \tag{767}$$
$$\cos(x) + (\cos(x) + \cos(\cos(x))) \tag{768}$$
$$\cos(x) + (\cos(x) + (2 + 2)) \tag{769}$$
$$\cos(x) + (\cos(x) + (2 + x)) \tag{770}$$
$$\cos(x) + (\cos(x) + (2 + \sin(x))) \tag{771}$$
$$\cos(x) + (\cos(x) + (2 + \cos(x))) \tag{772}$$
$$\cos(x) + (\cos(x) + (x + x)) \tag{773}$$

Differentiation (step 2)

$$\frac{d}{dx}\cos(x) + \frac{d}{dx}(\cos(x) + \cos(2)) \tag{765}$$

$$\frac{d}{dx}\cos(x) + \frac{d}{dx}(\cos(x) + \cos(x)) \tag{766}$$

$$\frac{d}{dx}\cos(x) + \frac{d}{dx}(\cos(x) + \cos(\sin(x))) \tag{767}$$

$$\frac{d}{dx}\cos(x) + \frac{d}{dx}(\cos(x) + \cos(\cos(x))) \tag{768}$$

$$\frac{d}{dx}\cos(x) + \frac{d}{dx}(\cos(x) + (2 + 2)) \tag{769}$$

$$\frac{d}{dx}\cos(x) + \frac{d}{dx}(\cos(x) + (2 + x)) \tag{770}$$

$$\frac{d}{dx}\cos(x) + \frac{d}{dx}(\cos(x) + (2 + \sin(x))) \tag{771}$$

Differentiation (step 3)

$$(-1) \cdot \sin(x) + \left(\frac{d}{dx} \cos(x) + \frac{d}{dx} \sin(x) \right) \tag{762}$$

$$(-1) \cdot \sin(x) + \left(\frac{d}{dx} \cos(x) + \frac{d}{dx} \sin(\sin(x)) \right) \tag{763}$$

$$(-1) \cdot \sin(x) + \left(\frac{d}{dx} \cos(x) + \frac{d}{dx} \sin(\cos(x)) \right) \tag{764}$$

$$(-1) \cdot \sin(x) + \left(\frac{d}{dx} \cos(x) + \frac{d}{dx} \cos(2) \right) \tag{765}$$

$$(-1) \cdot \sin(x) + \left(\frac{d}{dx} \cos(x) + \frac{d}{dx} \cos(x) \right) \tag{766}$$

$$(-1) \cdot \sin(x) + \left(\frac{d}{dx} \cos(x) + \frac{d}{dx} \cos(\sin(x)) \right) \tag{767}$$

Differentiation (step 4)

$$(-1) \cdot \sin(x) + \left((-1) \cdot \sin(x) + \left(\frac{d}{du} \cos(u) \right)_{u=\sin(x)} \cdot \frac{d}{dx} \sin(x) \right) \tag{767}$$

$$(-1) \cdot \sin(x) + \left((-1) \cdot \sin(x) + \left(\frac{d}{du} \cos(u) \right)_{u=\cos(x)} \cdot \frac{d}{dx} \cos(x) \right) \tag{768}$$

$$(-1) \cdot \sin(x) + \left((-1) \cdot \sin(x) + \left(\frac{d}{dx} 2 + \frac{d}{dx} 2 \right) \right) \tag{769}$$

$$(-1) \cdot \sin(x) + \left((-1) \cdot \sin(x) + \left(\frac{d}{dx} 2 + \frac{d}{dx} x \right) \right) \tag{770}$$

$$(-1) \cdot \sin(x) + \left((-1) \cdot \sin(x) + \left(\frac{d}{dx} 2 + \frac{d}{dx} \sin(x) \right) \right) \tag{771}$$

$$(-1) \cdot \sin(x) + \left((-1) \cdot \sin(x) + \left(\frac{d}{dx} 2 + \frac{d}{dx} \cos(x) \right) \right) \tag{772}$$

$$(-1) \cdot \sin(x) + \left((-1) \cdot \sin(x) + \left(\frac{d}{dx} x + \frac{d}{dx} x \right) \right) \tag{773}$$

Differentiation (step 5)

$$(-1) \cdot \sin(x) + ((-1) \cdot \sin(x) + \cos(\sin(x)) \cdot \cos(x)) \tag{763}$$

$$(-1) \cdot \sin(x) + ((-1) \cdot \sin(x) + \cos(\cos(x)) \cdot (-1) \cdot \sin(x)) \tag{764}$$

$$(-1) \cdot \sin(x) + ((-1) \cdot \sin(x) + 0 \cdot 0) \tag{765}$$

$$(-1) \cdot \sin(x) + ((-1) \cdot \sin(x) + (-1) \cdot \sin(x)) \tag{766}$$

$$(-1) \cdot \sin(x) + ((-1) \cdot \sin(x) + (-1) \cdot \sin(\sin(x)) \cdot \cos(x)) \tag{767}$$

$$(-1) \cdot \sin(x) + ((-1) \cdot \sin(x) + (-1) \cdot \sin(\cos(x)) \cdot (-1) \cdot \sin(x)) \tag{768}$$

$$(-1) \cdot \sin(x) + ((-1) \cdot \sin(x) + (0 + 0)) \tag{769}$$

$$(-1) \cdot \sin(x) + ((-1) \cdot \sin(x) + (0 + 1)) \tag{770}$$

$$(-1) \cdot \sin(x) + ((-1) \cdot \sin(x) + (0 + \cos(x))) \tag{771}$$

$$(-1) \cdot \sin(x) + ((-1) \cdot \sin(x) + (0 + (-1) \cdot \sin(x))) \tag{772}$$

$$(-1) \cdot \sin(x) + ((-1) \cdot \sin(x) + (1 + 1)) \tag{773}$$

Set theory

$$A \cup B = B \cup A$$

$$A \cap A^C = \emptyset$$

$$(A \cup B) \cap (B \cup A)^C = B \cap B^C$$

$$((A \cup B) \cap (B \cup A)^C) \cap ((A \cup B) \cap (B \cup A)^C)^C = \emptyset$$

$$etc.$$

Probability theory

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A|B)P(B) = P(A \cap B)$$

$$P(A \cup B) = P(A) + P(B) - P(A|B)P(B)$$

etc.

Analysis

$$\frac{d}{dx}\left(EA\frac{du}{dx}\right) + n = 0, u(0) = 0, u(L) = 0$$

$$\int_0^L \frac{d}{dx}\left(EA\frac{du}{dx}\right) vdx + \int_0^L nvdx = \int_0^L 0vdx, u(0) = 0, u(L) = 0$$

$$\int_0^L u\frac{dv}{dx}dx = \int_0^L \frac{du}{dx}vdx + u(0)v(L) - u(L)v(L)$$

$$\int_0^L \frac{d}{dx}\left(EA\frac{du}{dx}\right) vdx = \int_0^L EA\frac{du}{dx}\frac{dv}{dx}dx + EA\frac{du(0)}{dx}v(0) - EA\frac{du(L)}{dx}v($$

etc.

Algebra

Fundamental theorem of Galois theory.
Number of automorphisms equals the degree of the extension.

$$|Gal\,(K/F)| = [K : F]$$

Field extension K/F.
Degree of the extension $[K : F]$.
Number of automorphisms in the Galois extension $|Gal(K/F)|$.
For example, if we know that
$6 = [K : F] = \left[\mathbb{Q}\left(\sqrt[3]{2}, \sqrt{-3}\right) : \mathbb{Q}\right]$, then we know instantly
that $6 = |Aut\,(K/F)| = \left|Aut\left(\mathbb{Q}\left(\sqrt[3]{2}, \sqrt{-3}\right)/\mathbb{Q}\right)\right|$ etc.

Automated Programming


```
int f(int x)
{
        return x*x;
}
```

---

f(f(x))

```
int ff(int x)
{
        return (x*x)*(x*x);
}
```

What is Special About This Research?

All calculations are done in fully **autonomous** way.

Why autonomous calculations?

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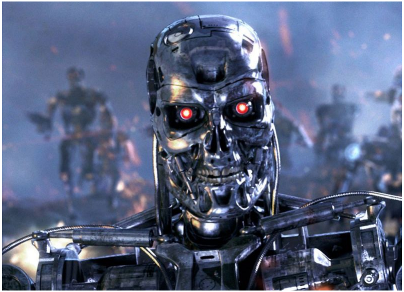
Why autonomous calculations?



Of course ... not this!

What is Special About This Research?

All calculations are done in fully **autonomous** way.

Why autonomous calculations?

Correctness
and
Scalability.

What is Special About This Research?

All calculations are done in fully **autonomous** way.

Why autonomous calculations?

Results are NOT Biased and Subjective .

Objectivity (Science)

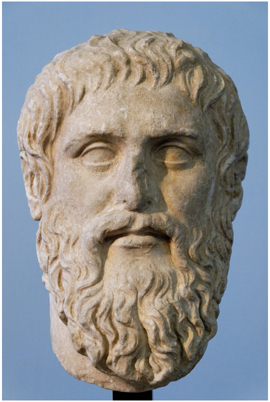
- **Objectivity** in science is an attempt to uncover truths about the natural world by eliminating personal biases, emotions, and false beliefs. .
- It is thus intimately related to the aim of testability and reproducibility.
- To be considered objective, the results of measurement must be communicated from person to person, and then demonstrated for third parties, as an advance in a collective understanding of the world. Such demonstrable knowledge has ordinarily conferred demonstrable **powers of prediction** or technology.

Possible Limitations

What are possible theoretical limitations of autonomous self-adaptive computational methods?

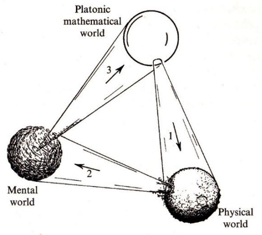
Platonic Idealism

- Platonic idealism is a form of metaphysical objectivism, holding that the **ideas exist independently from the individual**.



Mathematical Realism

- Mathematical entities exist **independently of the human mind**.
- Humans do not invent mathematics, but rather **discover it**, and any other intelligent beings in the universe would presumably do the same.



Logicism

- **Logicism** is the thesis that mathematics is reducible to logic, and hence nothing but a part of logic.
- Logicists hold that mathematics can be known a priori, but suggest that our knowledge of mathematics is just part of our knowledge of logic in general, and is thus analytic, not requiring any special faculty of mathematical intuition. In this view, logic is the proper foundation of mathematics, and all mathematical statements are necessary logical truths.

Logicism

Rudolf Carnap (1931) presents the logicist thesis in two parts:

- The concepts of mathematics can be derived from logical concepts through explicit definitions.
- The theorems of mathematics can be derived from logical axioms through purely logical deduction.

Hilbert's program.

- Hilbert proposed to ground all existing theories to a finite, complete set of axioms, and provide a proof that these axioms were consistent. Hilbert proposed that the consistency of more complicated systems, such as real analysis, could be proven in terms of simpler systems. Ultimately, the consistency of all of mathematics could be reduced to basic arithmetic.

Logicism

Gödel's incompleteness theorems:

- The first incompleteness theorem states that no consistent system of axioms whose theorems can be listed by an effective procedure (i.e., an algorithm) is capable of proving all truths about the arithmetic of the natural numbers.
- The second incompleteness theorem, an extension of the first, shows that the system cannot demonstrate its own consistency.

Tarski's undefinability theorem: arithmetical truth cannot be defined in arithmetic.

Possible Limitations

Due to Gödel's incompleteness theorems it is not possible to verify correctness of all conclusions generated by the system without interaction with the "outside world".

Formalism

- Formalism holds that mathematical statements may be thought of as statements about the consequences of certain string manipulation rules.
- According to formalism, mathematical truths are not about numbers and sets and triangles and the likein fact, they are not "about" **anything at all**.
- (Deductivism) Formalism need not mean that mathematics is nothing more than a meaningless symbolic game. It is usually hoped that there exists some interpretation in which the rules of the game hold.

Conventionalism

- Poincaré's use of non-Euclidean geometries in his work on differential equations convinced him that Euclidean geometry should not be regarded as a priori truth. He held that axioms in geometry should be chosen for the results they produce, not for their apparent coherence with human intuitions about the physical world.

Mathematical Intuitionism

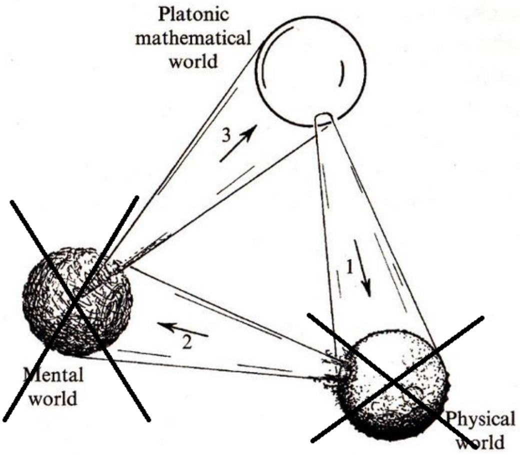
- In mathematics, intuitionism is a program of methodological reform whose motto is that "there are no non-experienced mathematical truths" (L. E. J. Brouwer).
- A major force behind intuitionism was L. E. J. Brouwer, who rejected the usefulness of formalized logic of any sort for mathematics. His student Arend Heyting postulated an intuitionistic logic, different from the classical Aristotelian logic; this logic does not contain the law of the excluded middle and therefore frowns upon proofs by contradiction. The axiom of choice is also rejected in most intuitionistic set theories, though in some versions it is accepted.

Other Theories

Etc.

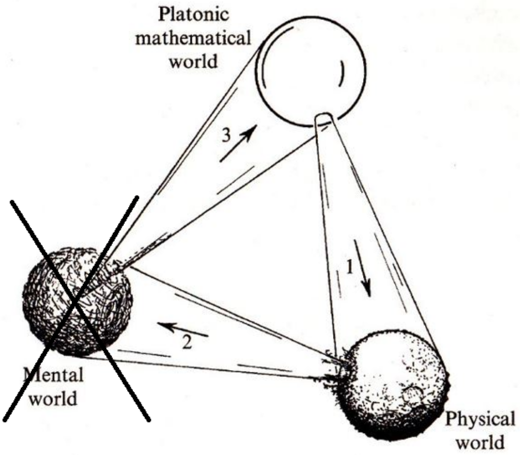
Self-Adaptive Computational Methods

Current version of the software.



Self-Adaptive Computational Methods

Future version of the software.



Solution of Algebraic Equations - - Deterministic Approach

- Linear equation $ax = b$
- Quadratic equation $ax^2 + bx + c = 0$
- Cubic equation $ax^3 + bx^2 + cx + d = 0$
- Non-elementary solutions $x \cdot e^x = 1$
- etc.

Appropriate method of solution can be applied to appropriate equation.

$$Known\ equation \rightarrow Method\ for\ solution$$

Solution of Algebraic Equations - - Machine Learning (Probabilistic Approach)

- Linear equation $ax = b$
- Quadratic equation $ax^2 + bx + c = 0$
- Cubic equation $ax^3 + bx^2 + cx + d = 0$
- Non-elementary solutions $x \cdot e^x = 1$
- etc.

Appropriate method of solution can be applied to appropriate equation.

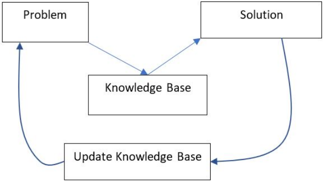
$$Known\ equation \rightarrow Method\ \ for\ \ solution$$

Equation is known "approximately" however it is a good initial choice.

Solution of Algebraic Equations - - Machine Learning

- It is possible to classify equations by using the neural networks.
- The initial guess provided by the neural networks can be used to find approximate solution of the problem.
- Training of the neural network can be done in fully autonomous way.
- The structure and features of the neural network can be crated in fully autonomous way.
- Initial guess provided by the probabilistic methods have to be verified in the future calculations.

Self-Adaptive Methods



Due to complexity of the problems the process of updating of the knowledge base never stops. The program runs practically indefinitely constantly improving his own computations. Calculations can be speed up by using parallel and distributed computing.

Conclusions

- Mathematical/scientific knowledge can be treated as independent units that can interact with each-other and create new, possibly useful knowledge.
- Generation of new knowledge can be fully automated and autonomous. No interaction with humans is necessary.
- Development of new knowledge is possible in many different fields (e.g. statistics, engineering, chemistry, biology, computer science etc.).

Conclusions (continued)

- By using presented methodology it is possible to create complex examples and appropriate computational methods relevant to many areas of mathematics as well as in other areas of science and engineering.
- Scientific results can be created in fully objective way without biased opinions of human researchers.
- By using self-adaptive computational methods it is possible to automatically generate new mathematical theorems without interactions with humans (consequently without human errors).
- Machine learning (the main computational method is NOT based on machine learning) can be used as source of good initial guess for processing mathematical information. Actually there is NO main computational method in the system.

Conclusions (continued)

- Once the information is available in the system it will NEVER be forgotten and can be used for generation of new mathematical theorems. From that perspective the system can be viewed as self-organizing archive of information.
- Development of this and similar systems should speed up cooperation among scientists around the world (theoretical possibility).

Conclusions (continued)

- Calculations can be done in distributed way (this option is experimental at this moment). Unlimited number of computers can process simultaneously in order to get the results faster. Calculations do not require existence of any centralized system.
- Turning off some computers slows down the calculation.
- Parallel computing can significantly speed up the calculations (future work).
- Autonomous interaction with external sources of information extends internal database of information and should increase productivity of the system (future work).

Conclusions (continued)

- Mathematical theorems can be used for the solution of many practical engineering and scientific problems if appropriate domain specific knowledge is available.
- I created some practical examples (civil engineering, oil engineering problems) by using presented system but not in fully autonomous way (work in progress).