Solution of the Wave Equation with Interval and Random Parameters

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Outline

Wave Equation

Wave equation

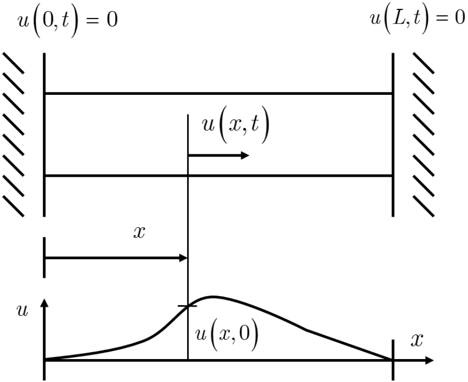
$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},$$

where $c = \sqrt{\dfrac{E}{\rho}}$.

Initial-boundary value problem for the wave equation

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \ (x, t) \in [0, L] \times [0, T] \\ u(0, t) = 0, \ t \in [0, T] \\ u(L, t) = 0, \ t \in [0, T] \\ u(x, 0) = u_0(x) \ x \in [0, L] \\ v(x, 0) = v_0(x) \ x \in [0, L] \end{cases}$$

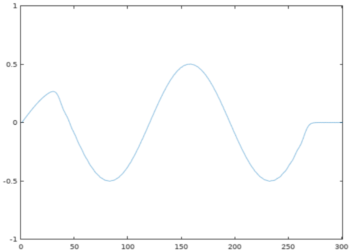
Initial-boundary value problem for the wave equation



Initial condition

Wave Equation

Typical solution of the wave equation.



Interval parameters (worst case analysis)

Solution of the equation with interval parameters for given (x, t) can be defined as the following set:

$$[\underline{u}(x,t), \overline{u}(x,t)] =$$

$$= \diamond\{u(x,t,p_1,...,p_m) : p_1 \in [\underline{p}_1, \overline{p}_1], ..., p_m \in [\underline{p}_m, \overline{p}_m]\}$$

where $[\underline{p}_1, \overline{p}_1], ..., [\underline{p}_m, \overline{p}_m]$ are interval parameters (for example E,ρ,n etc.) and $\diamond A$ is the smallest interval that contains the set A.

Function u is a solution of a PDE with the interval parameters

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Random parameters

Solution of the equation with random parameters $u(x, t, r(\omega))$ for given (x, t) can be defined as the function of random variable $r(\omega) = (r_1(\omega), ..., r_n(\omega))$.

Function u is a solution of a partial differential equation with random parameters (for example E,ρ,n etc.)

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Random and interval parameters

Solution of the equation with uncertain parameters $u(x, t, r(\omega), p)$ for given (x, t) can be defined as a function of random variable $r(\omega)$ and interval parameter p.

Function u is a solution of a partial differential equation with random and interval parameters (for example E, ρ, n etc.)

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Methods for the solution of the wave equation

- The Finite Difference Method
  - discretization of the second order differential equation
  - discretization of the first order differential equation
- The Finite Element Method
  - weak formulation
  - modal analysis
- Fourier Series

The Finite Difference Method

Differential equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Discretization

$$\frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{\Delta t^2} = c^2 \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{\Delta x^2}$$

$$u_{i,j+1} = 2u_{i,j} - u_{i,j-1} + \frac{c^2 \Delta t^2}{\Delta x^2} \left(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right)$$

The Finite Difference Method

Differential equation

$$\begin{cases} \frac{\partial u}{\partial t} = v \\ \frac{\partial v}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \end{cases}$$

Discretization

$$\begin{cases} u_{i,j+1} = u_{i,j} + v_{i,j}\Delta t \\ v_{i,j+1} = v_{i,j} + \frac{c^2 \Delta t}{\Delta x^2}(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}) \end{cases}$$

Weak formulation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$\int_0^L \frac{\partial^2 u}{\partial t^2} w dx = \int_0^L c^2 \frac{\partial^2 u}{\partial x^2} w dx$$

$$\int_0^L \frac{\partial^2 u}{\partial t^2} w dx + \int_0^L c^2 \frac{\partial u}{\partial x}\frac{\partial w}{\partial x} dx = 0$$

The Finite Element Method

Approximate solution

$$u = \sum_{i=1}^{n} u_i \varphi_i(x), \quad w = \sum_{j=1}^{n} v_j \varphi_j(x)$$

$$\frac{\partial u}{\partial x} = \sum_{i=1}^{n} u_i \frac{\partial \varphi_i(x)}{\partial x}$$

$$\frac{\partial w}{\partial x} = \sum_{j=1}^{n} w_j \frac{\partial \varphi_j(x)}{\partial x}$$

$$\frac{\partial^2 u}{\partial t^2} = \sum_{i=1}^{n} \ddot{u}_i \varphi_i(x)$$

The Finite Element Method

Approximate solution

$$\int_0^L \frac{\partial^2 u}{\partial t^2} v dx + \int_0^L c^2 \frac{\partial u}{\partial x}\frac{\partial w}{\partial x} dx = 0$$

$$\sum_{j=1}^{n} \left(\sum_{i=1}^{n} \int_0^L \varphi_i(x)\varphi_j(x) dx \ddot{u}_i + \right.$$

$$\left. + \sum_{i=1}^{n} \int_0^L c^2 \frac{\partial \varphi_i(x)}{\partial x}\frac{\partial \varphi_j(x)}{\partial x} dx \cdot u_i \right) w_j = 0$$

The Finite Element Method

Matrix form of the solution

$$M\ddot{u} + Ku = 0$$

where

$$M_{i,j} = \int_0^L \varphi_i(x)\varphi_j(x)dx$$

$$K_{i,j} = \int_0^L c^2 \frac{\partial \varphi_i(x)}{\partial x}\frac{\partial \varphi_j(x)}{\partial x}dx$$

The Finite Element Method

Modal analysis

$$M\ddot{u} + Ku = 0 \tag{1}$$

$$u = W\sin(\omega t + \varphi) = \begin{bmatrix} W_1 \\ W_2 \\ \dots \\ W_n \end{bmatrix} \sin(\omega t + \varphi)$$

$$\ddot{u} = -\omega^2 W\sin(\omega t + \varphi) = -\omega^2 \begin{bmatrix} W_1 \\ W_2 \\ \dots \\ W_n \end{bmatrix} \sin(\omega t + \varphi)$$

$$(K - \omega^2 M)W = 0$$

The Finite Element Method

General form of the solution:

$$u(t) = C_1 W_1 \sin(\omega_1 t + \varphi_1) + \ldots + C_n W_n \sin(\omega_n t + \varphi_n)$$

For example, for $L = 1, c = 1$ and two finite element the equation

$$\frac{L}{4 \cdot 6} \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} \ddot{u}_2 \\ \ddot{u}_3 \end{bmatrix} + \frac{4c^2}{L} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = C_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \sin\left(\sqrt{96}t + \varphi_1\right) +$$

$$+ C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \sin\left(\sqrt{\frac{96}{2}} t + \varphi_2\right)$$

Fourier Series

Differential equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Separation of variables

$$u(x,t) = y(x)\, w(t)$$

Let's substitute to the differential equation

$$\begin{gathered} \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \\ w''(t)\, y(x) = c^2 w(t)\, y''(x) \\ \frac{y''(x)}{y(x)} = \frac{1}{c^2} \frac{w''(t)}{w(t)} = -\lambda^2 \end{gathered}$$

Fourier Series

Differential equations for the functions y and w.

$$\begin{cases} \frac{y''(x)}{y(x)} = -\lambda^2 \\ \frac{1}{c^2}\frac{w''(t)}{w(t)} = -\lambda^2 \end{cases} \Rightarrow \begin{cases} y''(x) + \lambda^2 y(x) = 0 \\ w''(t) + \lambda^2 c^2 w(t) = 0 \end{cases}$$

Final form of the solution

$$u(x,t) = \sum_{n=0}^{\infty} \left(\frac{2}{L} \int_0^L f(x) \sin\left(\frac{\pi n x}{L}\right) dx \right) \cos\left(\frac{\pi n c t}{L}\right) \sin\left(\frac{\pi n x}{L}\right)$$

Upper and lower solution for problems with the interval parameters

Lower and upper solution $\underline{u}(x,t)$, $\overline{u}(x,t)$ can be calculated by using the optimization methods:

$$\underline{u}(x,t) =$$

$$= \min\{u(x,t,p_1,...,p_m) : p_1 \in [\underline{p}_1, \overline{p}_1], ..., p_m \in [\underline{p}_m, \overline{p}_m]\}$$

$$\overline{u}(x,t) =$$

$$= \max\{u(x,t,p_1,...,p_m) : p_1 \in [\underline{p}_1, \overline{p}_1], ..., p_m \in [\underline{p}_m, \overline{p}_m]\}$$

The Finite Difference Method

Explicit method

$$u_{i,j+1} = 2u_{i,j} - u_{i,j-1} + \frac{c^2 \Delta t^2}{\Delta x^2} \left(u_{i-1,j} - 2u_{i,j} + u_{i+1,j} \right)$$

Calculation of the gradient

$$\frac{\partial}{\partial p_k} u_{i,j+1} = 2 \frac{\partial}{\partial p_k} u_{i,j} - \frac{\partial}{\partial p_k} u_{i,j-1} +$$

$$+ \frac{\partial}{\partial p_k} \left(\frac{c^2 \Delta t^2}{\Delta x^2} \right) \left(u_{i-1,j} - 2u_{i,j} + u_{i+1,j} \right) +$$

$$+ \frac{c^2 \Delta t^2}{\Delta x^2} \left(\frac{\partial}{\partial p_k} u_{i-1,j} - 2 \frac{\partial}{\partial p_k} u_{i,j} + \frac{\partial}{\partial p_k} u_{i+1,j} \right)$$

The Finite Element Method

PDE after discretization

$$M\ddot{u} + Ku = 0$$

Calculation of the gradient

$$M\ddot{v} + Kv = -\frac{\partial}{\partial p_k}M\ddot{u} - \frac{\partial}{\partial p_k}Ku$$

where $v = \frac{\partial}{\partial p_k}u$.

Fourier Series

Solution

$$y(x,t) = \sum_{n=0}^{\infty} \left(\frac{2}{L} \int_0^L f(x) \sin\left(\frac{\pi n x}{L}\right) dx \right) \cos\left(\frac{\pi n c t}{L}\right) \sin\left(\frac{\pi n x}{L}\right)$$

Calculation of the gradient

$$\frac{\partial}{\partial p_k} y(x,t) =$$

$$= \sum_{n=0}^{\infty} \frac{\partial}{\partial p_k} \left(\left(\frac{2}{L} \int_0^L f(x) \sin\left(\frac{\pi n x}{L}\right) dx \right) \cos\left(\frac{\pi n c t}{L}\right) \sin\left(\frac{\pi n x}{L}\right) \right)$$

Steepest Descent Method

In order to find maximum/minimum of the function u it is possible to apply a modified version of the steepest descent algorithm.

1. Given x_0, set $k = 0$.
2. $d^k = -\nabla f(x_k)$. If $d^k = 0$ then stop.
3. Solve $min_\alpha f(x_k + \alpha d^k)$ for the step size α_k. If we know second derivative H then $\alpha_k = \frac{d_k^T d_k}{d_k^T H(x_k) d_k}$.
4. Set $x_{k+1} = x_k + \alpha_k d_k$, update $k = k + 1$. Go to step 1.

Parameter dependent probabilistic solution

For every specific value of the interval parameters $p_1, ..., p_m$ it is possible to calculate the probabilistic solution $u(x, t, \omega, p_1, ..., p_m)$.
Now it is possible to calculate extreme value of the probabilistic events. For example probability of failure. Probability of failure

$$P_f = P_\Omega\{\omega \in \Omega : g(u(\omega)) \leq 0\} \in [\underline{P}_f, \overline{P}_f]$$

Lower bound of the probability of failure

$$\underline{P}_f = \min\left\{P_\Omega\{\omega \in \Omega : g(u(\omega), p) \leq 0\} : p_i \in \left[\underline{p}_i, \overline{p}_i\right]\right\}$$

Upper bound of the probability of failure

$$\overline{P}_f = \max\left\{P_\Omega\{\omega \in \Omega : g(u(\omega), p) \leq 0 : p_i \in \left[\underline{p}_i, \overline{p}_i\right]\right\}$$

Upper and lower solution with random parameters

In the specific case when the upper and lower solution does not depend on the combination of random parameters for every specific $\omega \in \Omega$ it is possible to calculate upper and lower solution $\underline{u}(x,t,\omega), \overline{u}(x,t,\omega)$ and then calculate all kinds of probabilistic results by using for example Monte Carlo simulations. For example it is possible to calculate the probability that $u_{min} < u < u_{max}$ in the following way:

$$P_\Omega\{\omega \in \Omega : u_{min} < \underline{u}(x,t,\omega), \overline{u}(x,t,\omega) < u_{max}\} =$$

$$= P_\Omega\{\omega \in \Omega : [u_{min}, u_{max}] \supset \; [u(x,t,\omega), \overline{u}(x,t,\omega)]\}$$

where P_Ω is appropriate probability measure.

Upper and lower probability

Lower probability

$$Bel(A) = P_\Omega\{\omega \in \Omega : [\underline{u}(x,t,\omega), \overline{u}(x,t,\omega)] \subseteq A\}$$

Upper probability

$$Pl(A) = P_\Omega\{\omega \in \Omega : [\underline{u}(x,t,\omega), \overline{u}(x,t,\omega)] \cap A = 0\}$$

Reliability of engineering structures

Probability of failure

$$P_f = P_\Omega\{\omega \in \Omega : g(u(\omega)) \leq 0\} \in [\underline{P}_f, \overline{P}_f]$$

Lower probability

$$\underline{P}_f = P_\Omega\{\omega \in \Omega : g([\underline{u}(\omega), \overline{u}(\omega)]) \subseteq [0, \infty)\}$$

Upper probability

$$\overline{P}_f = P_\Omega\{\omega \in \Omega : g([\underline{u}(\omega), \overline{u}(\omega)]) \cap [0, \infty) \neq \emptyset\}$$

Example

Wave equation with uncertain initial conditions

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \ (x,t) \in (0,L) \times (0,T) \\ u(0,t) = 0, \ \ t \in [0,T] \\ u(L,t) = 0, \ \ t \in [0,T] \\ u(x,0) \in [\underline{u}_0(x), \overline{u}_0(x)] \ x \in [0,L] \\ v(x,0) = 0 \ x \in [0,L] \end{cases}$$

and c is a random parameter with some probability density function.

Example

Explicit method

$$u_{i,j+1} = 2u_{i,j} - u_{i,j-1} + \frac{c^2 \Delta t^2}{\Delta x^2}\left(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right)$$

Calculation of the gradient

$$u_{i,1} = u_{i,0} + v_{0,i,0}\Delta t = u_{i,0} \Rightarrow \frac{\partial}{\partial u_0} u_{i,1} = 1$$

$$\frac{\partial}{\partial u_0} u_{i,2} = 2\frac{\partial}{\partial u_0} u_{i,1} - \frac{\partial}{\partial u_0} u_{i,0} +$$
$$+ \frac{c^2 \Delta t^2}{\Delta x^2}\left(\frac{\partial}{\partial u_0} u_{i-1,1} - 2\frac{\partial}{\partial u_0} u_{i,1} + \frac{\partial}{\partial u_0} u_{i+1,1}\right) > 0$$

for t in some interval $[0, T]$.

Example

The interval solution

$$[\underline{u}(x,t,\omega), \overline{u}(x,t,\omega)] = [u(x,t,\omega,\underline{u}_0), u(x,t,\omega,\overline{u}_0)]$$

If c is a random parameter, then now it is possible to consider upper and lower probability of different events by using above described interval solution.

Conclusions

- Interval solution of the wave equation can be calculated by using different numerical methods and appropriate optimization algorithms. In this paper, 3 numerical methods and one optimization method were presented.
- If there are interval and random parameters it is possible to calculate upper and lower bound of the probability of different events.
- If the interval solution depend always on the same combination of parameters, then it is possible to compute upper and lower probabilistic solution and compute different probabilistic events by using this interval solution.