Finite Difference Equations with the Interval Parameters

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Outline

- Modeling of uncertainty
- Set valued solutions
- Automated theorem proving
Truss structure

Diagram of a truss structure with labeled nodes and forces. The structure is supported at both ends and loaded with forces $P_1$, $P_2$, and $P_3$. Nodes are numbered from 1 to 15, and the lengths are marked as $L$. The diagram includes diagonals and vertical supports.
Perturbated forces

\[ P = P_0 \pm \Delta P \]

5% uncertainty

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<td>10,00184</td>
<td>10,00126</td>
<td>60,18381</td>
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\[ P_1 \]

\[ P_2 \]

\[ P_3 \]
Example II (model)
Example II (results)

uncertainty = 5% min= 82711 max= 188847
More examples
More examples

Dr. Eng. Andrzej Pownuk

[ Publications ] [ Professional biography ]
[ Research interests ] [ Interval equations (list of references) ] [ Teaching ] [ For students ] [ On-line Advisor ]
[ Examples of applications ] [ Interval FEM on Wikipedia ] [ Programming Experience ]
[ Computational Science Ph.D. Program at UTEP ] [ Tools ]
[ My schedule ] [ CPS ]

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Second order approximation

\[ u' = (k/m) * u = (P/m) * \cos(w*t) \]

\[ v = u' \]

Calculation

- \( n_0 = 1 \)
- \( v_0 = 1 \)
- \( P = 1 \)
- \( k_{\text{Min}} = 1 \)
- \( k_{\text{Max}} = 2 \)
- \( m_{\text{Min}} = 1 \)
- \( m_{\text{Max}} = 2 \)
- \( w = 2 \)
- number of intermediate points = 5
- \( dt = 0.05 \)
- number of timesteps = 100

Graph showing two curves.
2D PDE

\[
\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + T = f
\]

\[
\frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{\Delta x^2} + \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{\Delta y^2} + T_{i,j} = f_{i,j}
\]
Solution of 2D heat equation
3D Heat transfer problem

\[ \alpha \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + q = \frac{\partial T}{\partial t} \]

\[ \alpha \left( \frac{T_{i+1,j,k} - 2T_{i,j,k} + T_{i-1,j,k}}{\Delta x^2} + \frac{T_{i,j+1,k} - 2T_{i,j,k} + T_{i,j-1,k}}{\Delta y^2} + \frac{T_{i,j,k+1} - 2T_{i,j,k} + T_{i,j,k-1}}{\Delta z^2} \right) + q_{i,j,k} = \frac{T_{i,j,k}^{t+\Delta t} - T_{i,j,k}^t}{\Delta t} \]

\[ T_{i,j,k}^{t+\Delta t} = T_{i,j,k}^t + \alpha \left( \frac{T_{i+1,j,k} - 2T_{i,j,k} + T_{i-1,j,k}}{\Delta x^2} + \frac{T_{i,j+1,k} - 2T_{i,j,k} + T_{i,j-1,k}}{\Delta y^2} + \frac{T_{i,j,k+1} - 2T_{i,j,k} + T_{i,j,k-1}}{\Delta z^2} \right) \Delta t + q_{i,j,k} \Delta t \]
Engineering simulations
Uncertain parameters:

\[ ax = b \quad x = \frac{b}{a} \]

Example: \([1, 2]x = [1, 4]\)

\[ x = \frac{[1, 4]}{[1, 2]} = [1, 2] \]
Extreme values of montone functions

\[ f(x) = x^2, \quad x \in [1, 2] \]

\[ \frac{df(x)}{dx} = 2x \in [2, 4] \]

\[ f_\bar = f(\bar x) = f(1) = 1^2 = 1 \]

\[ f = f(\bar x) = f(2) = 2^2 = 4 \]

\[ f(x) \in [f_\bar, f] = [1, 4] \]
Interval solution of PDE

\[ u(x,p) = \{ u(x,p) : p \in \mathbb{P} \} \]
Set valued solution of ODE
Random solution of ODE
Set valued solution of ODE
Solution set of algebraic equations
Set valued solution of PDE
Set valued solution of PDE
Solution set of the equations with the interval parameters is not an interval - it is not necessary to reduce the set-valued solution the interval solution.

\[ u(p) = \Box u(p) \]

- it is possible to use the set-valued solution directly in calculations.
Finite difference method

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f
\]

\[
\left( \frac{\partial^2 u}{\partial x^2} \right)_{ij} + \left( \frac{\partial^2 u}{\partial y^2} \right)_{ij} = f_{ij}
\]

\[
\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2} = f_{ij}
\]

\[
A(p)u = b(p)
\]
Gradient method

\[ A(p) \frac{\partial u}{\partial p} = \frac{\partial b(p)}{\partial p} - \frac{\partial A(p)}{\partial p} u \]

\[ u_i = \begin{cases} 
  m \text{ in } u_i \\
  A(p)u = b(p), \quad \bar{u}_i = \begin{cases}
  m \text{ ax } u_i \\
  A(p)u = b(p) \\
  p \in p
\end{cases}
\end{cases} \]

\[ p \in p \]
Optimization

\[ p^{m_{\text{in}}} = \arg \min_{u_i} \left\{ \begin{array}{l}
A(p)u = b(p) \\
p \in \mathbf{p}
\end{array} \right\} \]

\[ p^{m_{\text{ax}}} = \arg \max_{u_i} \left\{ \begin{array}{l}
A(p)u = b(p) \\
p \in \mathbf{p}
\end{array} \right\} \]
New solution procedure

Stokes theorem

\[ \int_{\Sigma} \nabla \times F \cdot d\Sigma = \oint_{\partial\Sigma} F \cdot dr, \]

Butterfly lemma or Zassenhaus lemma

Lemma: Suppose \((G, \Omega)\) is a group with operators and \(A\) and \(C\) are subgroups. Suppose \(B \triangleleft A\) and \(D \triangleleft C\) are stable subgroups. Then,

\[ (A \cap C)'B/(A \cap D)B \text{ is isomorphic to } (A \cap C)'D/(B \cap C)'D. \]
It is possible to use mathematical theorems in an automatic/computational way without human presence. (automated theorem proving)
It is possible to apply automatic theorem proving to the solution of the interval equations.
Automated theorem proving can be applied to the solution any scientific field. (biology, chemistry, physics …)

however this is not the topic of my research
look for example at: www.cyc.com
theory of everything …
It is possible to build a system who is able to use its own results in order to improve itself (self learning).
Examples

- $2*[1,2]*x - x = 1$
- $(2*[1,2] - 1)x = 1$
- $([2,4] - 1)x = 1$
- $[1,3]x = 1$
- $x = 1/[1,3]$

The system is able to show all intermediate steps of the calculations.

Look also at:
http://www.wolframalpha.com
Example

\[ \frac{du}{dx} + up = 0, u(0) = 0, p \in [1,2] \]

\[ \frac{u_{i+1} - u_i}{\Delta x} + u_i p = 0, u_0 = 0, p \in [1,2] \]

\[ u_{i+1} = u_i p \Delta x + u_i, u_0 = 0, p \in [1,2] \]

\[ \underline{u}_i = \min \{ u_i : u_{i+1} = u_i p \Delta x + u_i, u_0 = 0, p \in [1,2] \} \]

\[ \overline{u}_i = \max \{ u_i : u_{i+1} = u_i p \Delta x + u_i, u_0 = 0, p \in [1,2] \} \]
Example

- cats like food
- food=[milk< canned food<mice]
  (example problem with the interval data)

- What kind of food do the cats like most?

- Answer: mice
Conclusions

- It is possible to solve efficiently the finite difference equations of computational mechanics with the interval parameters.
- In many cases it is possible to use general solution set of the system of interval equations instead of replacing it by the interval solution set.
- It is possible to apply automatic theorem proving in order to solve many problems in the theory of interval equations.
- It is possible to create self improving computational systems.