MODELLING OF HEAT TRANSFER IN BIOLOGICAL TISSUE BY INTERVAL FEM

Marek Jasiński
Department for Strength of Materials and Computational Mechanics
Silesian University of Technology
44-100 Gliwice ul. Konarskiego 18a
Andrzej Pownuk
Department of Theoretical Mechanics
Silesian University of Technology
ul. Krzywoustego 7, 44-100 Gliwice, Poland

In this paper, an algorithm of calculation of extreme values of temperature based on interval arithmetic is presented. Many mechanical systems with uncertain parameters $\lambda \in \mathbf{\Lambda}$ can be described by parameter dependent system of linear equations $K(\lambda)T=B(\lambda)$. Using natural interval extension of real function, one can transform system of linear equation into system of linear interval equation $\boldsymbol{K}(\mathbf{\Lambda})\boldsymbol{T}=\boldsymbol{B}(\mathbf{\Lambda})$. Solution of system of linear interval equations always contains the exact solution of parameter dependent system of equation. This technique is called Interval Finite Element Method. New method of computation of extreme values of mechanical quantities based on monotonicity test is introduced. This method can give exact solution of parameter dependent system of equations.

1 Introduction

The study of the influence of parameters upon behaviour of mathematical models is one of the basic problems of computational mechanics. Usually, one is interested in systems which are locally stable in the sense that their qualitative behaviour does not change under small variations of the parameters. Here, some form of perturbation theory may be the appropriate tool [36]. One often needs to know explicitly the properties of the solutions for a large region of physical significance. The mathematical models for describing the systems considered in this paper are in the form [29, 36]

$$F(z,t)=0 \tag{1}$$

where $F : Z \times T \rightarrow Z$. Space z characterizes the state of the system, t denotes the parameter variable allowed to vary in space T.

Consider now mechanical system (1) with uncertain parameters t. If sufficient experimental information are available, probablistic methods should be applied [40, 41]. Alternatively convex model of uncertainty should be applied [2, 5, 8, 43].

One of the simplest ways of representation of uncertain or inexact data, as well as inexact computations with them, is based on interval arithmetic [1, 22, 26, 29, 42]. Other methods are based on set valued analysis [38] and classical theory of optymization [2, 5, 7, 9, 12, 23, 41, 44]. Convex model of uncertainty can be represented also by ellipse [10, 32, 43].

When function is sufficiently smooth to calculation extreme values, Kuhn-Tucker condition [11] can be used. When function $z(t)$ is given explicitly to calculation extreme value, one can use interval global optimization [39] or other global optimization method (solon.cma.univie.ac.at/~neum/glopt).

2 Interval arithmetic

A real interval is a set of real numbers such that

$$\mathbf{x} = [\underline{x}, \overline{x}] = \{x \in R : \underline{x} \le x \le \overline{x}\} \quad \textbf{(2)}$$

The set of all intervals is denoted by *IR* [1, 29] and called a real interval space. Operations and functions on reals are naturally extended to interval operands according to the general formulas [26]

$$\mathbf{x} \oplus \mathbf{y} = \{x \oplus y : x \in \mathbf{x}, y \in \mathbf{y}, \oplus \in \{+,-,\cdot,/\}\} \quad \textbf{(3)}$$

$$f(\mathbf{x}_1,\ldots,\mathbf{x}_n) = \{f(x_1,\ldots,x_n) : x_1 \in \mathbf{x}_1, \ldots, x_n \in \mathbf{x}_n\} \quad (4)$$

The function *f*(.) is called *programmable* if *f*(*x*) can be built up from the arithmetic, the logical and the comparison operators and some collection of standard transcendental functions (like *sin, cos, power*, etc.). Especially, given an argument *x*, the function value *f*(*x*) can be computed with a finite number of operations [29]. All the functions in this paper are assumed to be programmable.

Another important property of arithmetic operations on intervals is called *inclusion isotonicity*

$$(\mathbf{a} \subseteq \mathbf{c}) \wedge (\mathbf{b} \subseteq \mathbf{d}) \Rightarrow \mathbf{a} \oplus \mathbf{b} \subseteq \mathbf{c} \oplus \mathbf{d} \quad (5)$$

the result of straightforward calculation of interval expression will always include the proper result ($\oplus$ is any interval arithmetic operations).

Let $\mathbf{x} \in IR$, then the natural interval extension of programmable function *f*(.) to **x** is defined as expression which is obtained from the expression *f*(*x*) by replacing each occurrence of the variable *x* by the box **x**, the arithmetic operations of *R* by the corresponding interval arithmetic operations, and each occurrence of pre-declared function *g*(.) by the corresponding inclusion function ***g*(.)** [26]. From the inclusion isotonicity of interval arithmetic operations [1, 26], and from the properties of the pre-declared inclusions i.e. the $\boldsymbol{f}(\,.\,)$'s, it follows that every inclusion function ***f*(x)** has a property

$$x \in \mathbf{x} \text{ implies } f(x) \in \boldsymbol{f}(\mathbf{x}) \quad (6)$$

Property (6) is the key to almost all interval arithmetic applications and results E.Hansen suggested at an international interval arithmetic workshop in Colombus, Ohio, September 1987, that (6) should be called the *fundamental property of interval arithmetic*. For any bounded set of real numbers S, one can define a *smallest interval enclosure* of the set [22]

$$hull\ S = [inf\ S, sup\ S] \quad (7)$$

3 Systems of linear interval equations

Let us consider a linear interval system of equations with an interval coefficient matrix $\mathbf{A} \in IR^{n \times n}$ and an interval right-hand vector $\mathbf{B} \in IR^{n}$ [29]

$$\mathbf{A}X = \mathbf{B} \quad (8)$$

The solution set of Eq. (8) is defined as [29]

$$\sum(\mathbf{A},\mathbf{B}) = \{X \in R^n : \exists A \in \mathbf{A}, \exists B \in \mathbf{B}, A \cdot X = B\} \quad (9)$$

Calculating (and representing) the solutions set $\sum(\mathbf{A},\mathbf{B})$ may be quite hard and impractical, especially for larger *n* [22]. Therefore, for many practical purposes, one is satisfied with the *interval enclosure* of the solution set (9). The smallest enclosure is the *hull* of the set [22, 29]

$$hull \sum(\mathbf{A},\mathbf{B}) = \left[inf \sum(\mathbf{A},\mathbf{B}), sup \sum(\mathbf{A},\mathbf{B})\right] \quad (10)$$

There are many methods for solution of equation (8). The simplest method is the use of all combinations of the endpoints of intervals of matrix **A** and vector **B** [13, 22]. Others like Rohn sign algorithm [37] or linear programming method [19, 22] are based on the theorem of Oettli and Prager [29]. Some methods give only interval estimation of set [22, 29]. Computation of exact solution set (9 or 10) is NP-hard [21].

4 Interval FEM

In this paper extreme values of temperature in biological tissue using Interval Finite Element Method [19, 20, 22, 27, 28, 33, 34] are calculated. The Pennes equation describing the steady temperature field in biological tissue is considered [14]

$$div[\lambda\, grad\, T(x)] + Q_{met} + Q_{perf} = 0 \tag{11}$$

Where λ is the thermal conductivity of tissue, Q_{met} is the metabolic heat source, Q_{perf} is the perfusion heat source, T is the temperature. Equation (11) is supplemented by boundary conditions which can be written in form

$$x \in \Gamma : \ \Phi\left[T(x), \frac{\partial T(x)}{\partial n} \right] = 0 \tag{12}$$

The perfusion heat source is as follows

$$Q_{perf}(x) = c_b \rho_b G_b [T(x) - T_b] \tag{13}$$

where c_b, ρ_b are the specific heat and mass density of blood, T_b is the temperature of blood and G_b is the perfusion rate. In this paper, the 1D problem is considered. In order to obtain the solution, the FEM is applied. Finite Element Method leads to the following system of equation

$$K(\lambda)T = B(\lambda) \tag{14}$$

where K is global heat conductivity matrix, T is the vector of unknown temperatures and B is the vector containing the information about boundary conditions, λ is an uncertain parameter. The exact solution set of (14) can be described as

$$\{(T_1(\lambda), ..., T_n(\lambda)) : \lambda \in \mathbf{\Lambda}\} = \{(T(x_1, \lambda), \ldots, T(x_n, \lambda)) : \lambda \in \mathbf{\Lambda}\} \tag{15}$$

Interval analysis provides a rigorous and realistic sensitivity analysis of the solution of (15) under perturbations of specified magnitude (i.e. not only asymptotically for „sufficient small" perturbations). Computation of exact solution set (15) is very difficult [15, 18, 29-31]. Uncertainty is introduced by assigning an interval value parameters using their interval extension [1]. Then the system of equation (14) become system of interval equations in form

$$\boldsymbol{K}(\mathbf{\Lambda})T = \boldsymbol{B}(\mathbf{\Lambda}) \tag{16}$$

From fundamental property of interval arithmetic [29], it arises that

$$\{(T_1(\lambda), ..., T_n(\lambda)) : \lambda \in \mathbf{\Lambda}\} = \{(T(x_1, \lambda), \ldots, T(x_n, \lambda)) : \lambda \in \mathbf{\Lambda}\} \subseteq \sum(\boldsymbol{K}(\mathbf{\Lambda}), \boldsymbol{B}(\mathbf{\Lambda})) \tag{17}$$

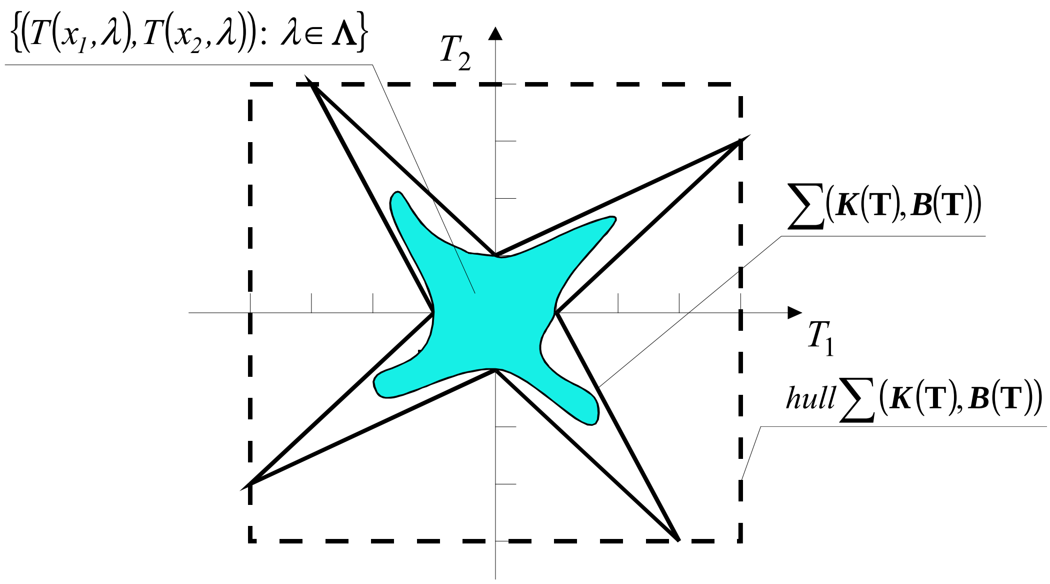

Fig.1 Solutions sets of equation (14) and (16).

i.e. solution of system of linear interval equation (16) always contain the exact solution set (15) in a nodal point x_i (i=1,...,n). Both solution sets (15, 17) are very complicated (see Fig.1). Because of this, in application, one use the smallest interval which contain the exact solution set (15 or 17) [22].

$$hull \sum \left(\boldsymbol{K}(\boldsymbol{\Lambda}), \boldsymbol{B}(\boldsymbol{\Lambda})\right) \in IR^n \tag{18}$$

Temperature field in biological tissue satisfy the following equations

$$\begin{cases} R_1 < r < R_2 : \dfrac{1}{r}\dfrac{d}{dr}\left(r\lambda \dfrac{dT(r)}{dr}\right) + Q = 0 \\ r = R_1 \quad : -\lambda \dfrac{dT(r)}{dr} = \alpha\left(T(r) - T_b\right) \\ r = R_2 \quad : T(r) = T_t \end{cases} \tag{19}$$

R_1, R_2 - internal and external radiuses of domain
α - heat transfer coefficient
T_t - tissue temperature
$Q = Q_{met} + Q_{perf}$ - constant value

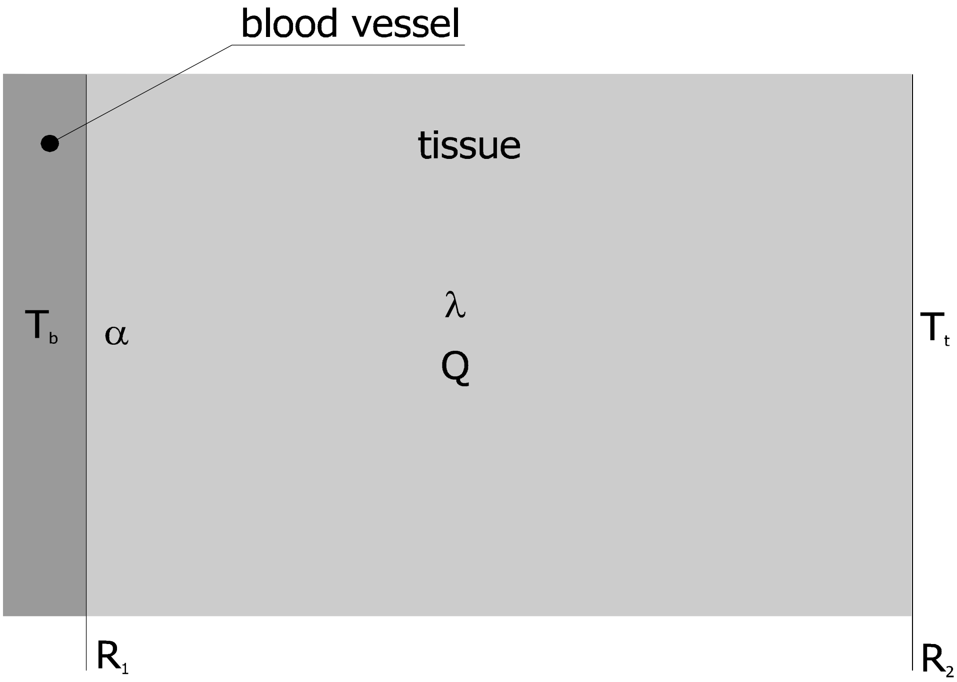

Fig. 2 The domain considered

Using the weighted residual method criterion one has

$$\int_{R_1}^{R_2} \left[\frac{d}{dr}\left(r\lambda \frac{dT}{dr}\right) + rQ\right] w(r) dr = 0 \tag{20}$$

and after integrating the first component of equation above by parts, one obtain the following equation

$$r\lambda \frac{dT}{dr} w(r) \bigg|_{R_1}^{R_2} - \int_{R_1}^{R_2} r\lambda \frac{dT}{dr}\frac{dw(r)}{dr} dr + \int_{R_1}^{R_2} r\, Q\, w(r) dr = 0 \tag{21}$$

The integrals from R_1 to R_2 are substituted by the sum of integrals

$$\left[r\lambda \frac{dT}{dr} w(r)\right]_{R_2} - \left[r\lambda \frac{dT}{dr} w(r)\right]_{R_1} - \sum_{\substack{(e)\\ i=1}}^{n} \int_{r_{i-1}}^{r_i} r\lambda \frac{dT}{dr}\frac{dw(r)}{dr} dr + \sum_{\substack{(e)\\ i=1}}^{n} \int_{r_{i-1}}^{r_i} r\, Q\, w(r) dr = 0 \tag{22}$$

Temperature at the finite element sub–domain is described by the linear function

$$r \in [r_{i-1}, r_i]: T(r) = \frac{r_i - r}{h} T_{i-1} + \frac{r - r_{i-1}}{h} T_i = N_{i-1} T_{i-1} + N_i T_i \quad (23)$$

After combination of equation (22), one obtain system of linear equation in the form

$$K(\lambda) T = B(\lambda) \quad (24)$$

Coefficients of global heat conductivity matrix for i=1, ... ,n-1

$$k_{0,0} = r_1^2 - r_0^2,\ k_{0,1} = r_0^2 - r_1^2 \quad (25)$$

$$k_{i,i} = r_{i+1}^2 - r_{i-1}^2,\ k_{i,i-1} = r_{i-1}^2 - r_i^2,\ k_{i,i+1} = r_i^2 - r_{i+1}^2 \quad (26)$$

$$k_{n,n-1} = r_{n-1}^2 - r_n^2,\ k_{n,n} = r_n^2 - r_{n-1}^2 \quad (27)$$

Coefficients of vector B for i=1, ... ,n-1

$$p_i = 4r_i^3 + r_{i-1}^3 + r_{i+1}^3 - 3r_i^2 (r_{i-1} - r_{i+1}) \quad (28)$$

and terms 0 and n

$$p_0 = r_1^3 + r_0^2 (2r_0 - 3r_1) + \frac{3R_1 \alpha T_b \lambda}{Qh} \quad (29)$$

$$p_n = \frac{3T_t \lambda}{Qh} \quad (30)$$

Finally, terms i=0, ... ,n

$$B = \frac{Qh}{3\lambda} P \quad (31)$$

One assumed that R_1=0.0005 [m], R_2=10*R1, α=2000, T_b= 32 [^oC], T_t=37 [^oC], Q=10245 [W/m^3], $\lambda \in$ [0.21,0.23] [W/mK]. The results of calculations are shown in Fig.3.

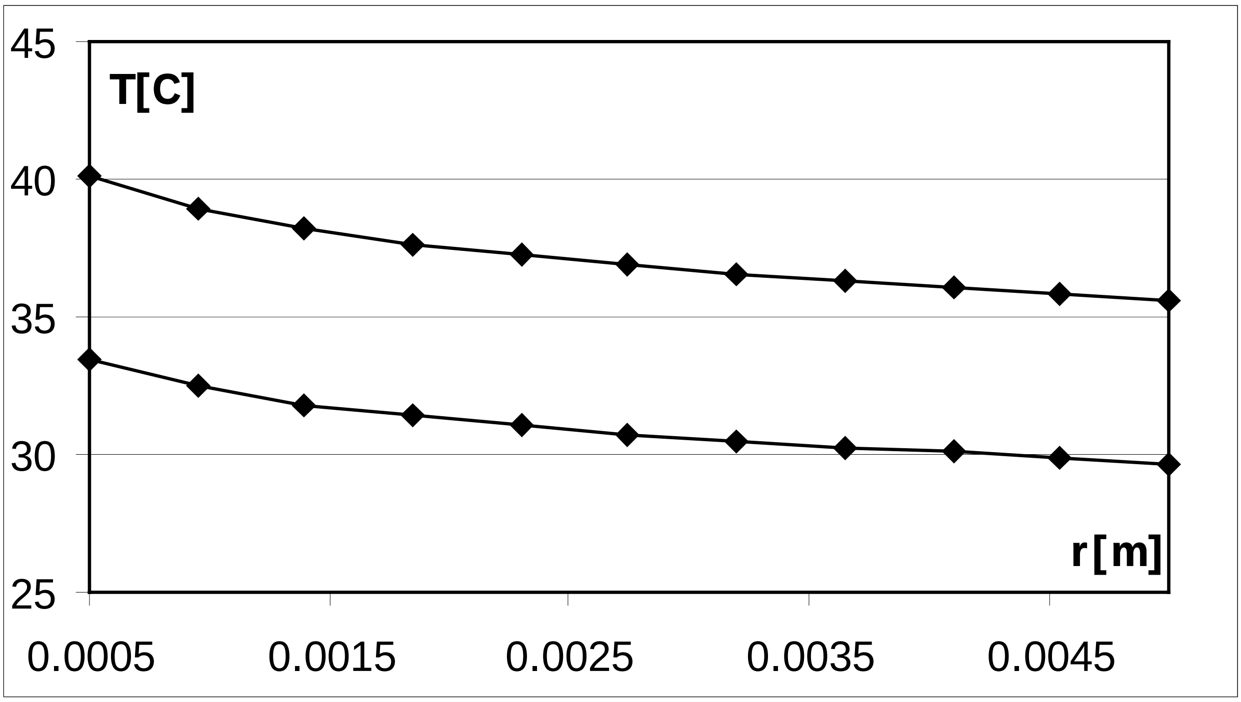

Fig. 3 Extreme values of temperature in biological tissue

5 Interval FEM based on monotonicity test

In many engineering problems, solution $T_i(\lambda)$ is monotone in $\mathbf{\Lambda}$. In this case extreme values of mechanical quantities one can calculate using endpoints of the interval $\mathbf{\Lambda}$

$$\underline{T}_i = min\{T_i(\underline{\lambda}), T_i(\overline{\lambda})\}, \quad \overline{T}_i = max\{T_i(\underline{\lambda}), T_i(\overline{\lambda})\} \tag{32}$$

From fundamental property of interval arithmetic (6), it arises that

$$\text{if } 0 \notin \frac{\partial \boldsymbol{T}_i(\mathbf{\Lambda})}{\partial \lambda} \text{ then } T_i(\lambda) \text{ is monotone in } \mathbf{\Lambda} \tag{33}$$

Derivative of temperature T_i, can be calculated using the implicit function theorem

$$K\frac{\partial T}{\partial \lambda} = \frac{\partial B}{\partial \lambda} - \frac{\partial K}{\partial \lambda}T \tag{34}$$

Interval extension of derivative (33), can be calculated as a solution of the following system of linear interval equations

$$\boldsymbol{K}(\mathbf{\Lambda})\frac{\partial T}{\partial \lambda} = \frac{\partial \boldsymbol{B}(\mathbf{\Lambda})}{\partial \lambda} - \frac{\partial \boldsymbol{K}(\mathbf{\Lambda})}{\partial \lambda}\boldsymbol{T}(\mathbf{\Lambda}) \tag{35}$$

where

$$\boldsymbol{T}(\mathbf{\Lambda}) = hull\sum(\boldsymbol{K}(\mathbf{\Lambda}), \boldsymbol{B}(\mathbf{\Lambda})) \tag{36}$$

From fundamental property of interval arithmetic (6), equation (35) and conclusion (33), it is arises that if

$$0 \notin \frac{\partial \boldsymbol{T}(\mathbf{\Lambda})}{\partial \lambda} = \sum\left(\boldsymbol{K}(\mathbf{\Lambda}), \frac{\partial \boldsymbol{B}(\mathbf{\Lambda})}{\partial \lambda} - \frac{\partial \boldsymbol{K}(\mathbf{\Lambda})}{\partial \lambda}\boldsymbol{T}(\mathbf{\Lambda})\right) \tag{37}$$

then functions $T_i(\lambda)$ are monotone in $\mathbf{\Lambda}$. If interval $\mathbf{\Lambda}$ is too large one can divide it into part $\mathbf{\Lambda}_i$ such that $\mathbf{\Lambda} = \bigcup_i \mathbf{\Lambda}_i$ and $int(\mathbf{\Lambda}_i) \cap int(\mathbf{\Lambda}_j) = \varnothing$ for $i \neq j$. If functions $T_i(\lambda)$ are monotone in all parts $\mathbf{\Lambda}_i$ then functions $T_i(\lambda)$ are monotone in $\mathbf{\Lambda}$.

Consider now the same boundary value problem (19) and initial data. The results of the calculations are shown in table 1. In the calculation, the interval $\mathbf{\Lambda}$ was divided into 40 parts $\mathbf{\Lambda}_i$.

Table 1 Results of calculation using Interval Finite Element Method with monotonicity test

	T_0	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}
$\underline{T}_i$ [°C]	36.586	35.470	34.782	34.284	33.894	33.573	33.302	33.065	32.857	32.669	32.500
$\overline{T}_i$ [°C]	36.619	35.494	34.800	34.298	33.905	33.582	33.308	33.070	32.859	32.671	32.500

6 Conclusions

The Interval Finite Element Method (IFEM) always gives solution in the form [19, 22]

$$\boldsymbol{T}(\mathbf{\Lambda}) = hull\sum(\boldsymbol{K}(\mathbf{\Lambda}), \boldsymbol{B}(\mathbf{\Lambda})) \tag{38}$$

This interval always contains exact solution set of parameter dependent system of equation (24). Both solutions are very similar only if width of interval $\mathbf{\Lambda}$ is very small. In other cases, the so called error explosion problem [17, 22] causes that Interval FEM gives extremely overestimated results [22]. Coefficient dependence problem provides the main source of overestimation [22]. This error is an integral part of IFEM and it is impossible to avoid this effect in algorithm which was presented in section 4. IFEM with monotonicity test gives exact solution of equation (24). One can applied this algorithm in a situation where parameter dependent solution is monotone.

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