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Applications of Higher Order Monotonicity Tests to the Solution of Equations with the Interval Parameters

Non-probabilistic error analysis

$$y = f(p_1, \ldots, p_n)$$

$$p_i \in \left[\underline{p}_i, \overline{p}_i \right]$$

$$y \in \left[\underline{y}, \overline{y} \right] = \left\{ y\left(p_1, \ldots, p_m\right) : p_i \in \left[\underline{p}_i, \overline{p}_i \right] \right\}$$

Example

$$t \in [0.5, 2][s]$$

$$s \in \frac{gt^2}{2} = [1.23, 19.6][m]$$

Taylor expansion

$$y(p_1,\ldots,p_m) \approx y(p_{10},\ldots,p_{m0}) + \frac{\partial y}{\partial p_1}(p_1 - p_{10}) + \ldots + \frac{\partial y}{\partial p_m}(p_m - p_{m0})$$

$$y_0 = y(p_{10},\ldots,p_{m0})$$

$$\Delta y \approx \left|\frac{\partial y}{\partial p_1}\right| \Delta p_1 + \ldots + \left|\frac{\partial y}{\partial p_m}\right| \Delta p_m$$

$$y \in \left[\underline{y}, \overline{y}\right] \approx [y_0 - \Delta y, y_0 + \Delta y]$$

Example

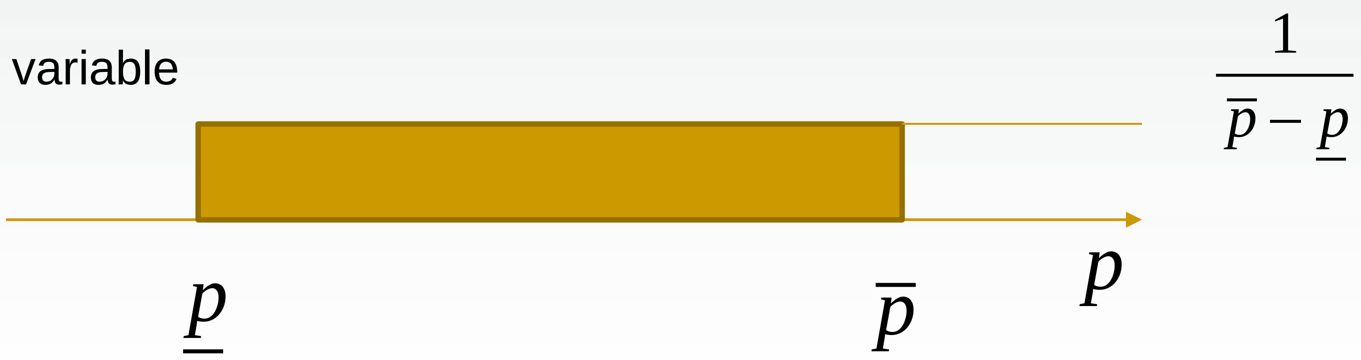
$$\Delta s = gt\Delta t$$

$$s \in [s_0 - \Delta s, s_0 + \Delta s] = \left[-1.5, 16.8\right][m]$$

Differences between intervals and uniformly distributed random variables

Interval

Random variable



Differences between intervals and uniformly distributed random variables

Intervals

$$\mathbf{p}_1 + \ldots + \mathbf{p}_m = \left[\sum_i \underline{p}_i , \sum_i \overline{p}_i \right] = n\left[\underline{p}, \overline{p} \right]$$

$$width\left(n\mathbf{p}\right) = n\left(\overline{p} - \underline{p}\right) = n \cdot width\left(\mathbf{p}\right)$$

Random variables

$$\sigma_{n\mathbf{p}} = \sqrt{\sum_i \sigma_i^2} = \sqrt{n\sigma^2} = \sigma\sqrt{n}$$

$$width\left(n\mathbf{p}\right) = \sqrt{n} \cdot width\left(\mathbf{p}\right)$$

Differences between intervals and uniformly distributed random variables

$$\frac{width(n\mathbf{p}_{\text{int}})}{width(n\mathbf{p}_{rand})} = \frac{n \cdot width(\mathbf{p}_{\text{int}})}{\sqrt{n} \cdot width(\mathbf{p}_{rand})} = \sqrt{n}$$

Example n=100

$$\frac{width(n\mathbf{p}_{\text{int}})}{width(n\mathbf{p}_{rand})} = \frac{n \cdot width(\mathbf{p}_{\text{int}})}{\sqrt{n} \cdot width(\mathbf{p}_{rand})} = \sqrt{100} = 10$$

Equations of continuum mechanics

Equation

$$A(x,u) = b(x)$$

Solution

$$u = u(x)$$

Equation (parameter dependent)

$$A(x,u,p) = b(x,p)$$

Solution (parameter dependent)

$$u = u(x,p)$$

References

http://andrzej.pownuk.com/IntervalEquations.htm

http://www.cs.utep.edu/interval-comp/books.html

List of researchers applying interval computations
http://www.cs.utep.edu/interval-comp/icpersons.html

Several definitions of the solution set

Interval equations

$$ax = b$$

Example

$$[1, 2]x = [1, 4]$$

$$x = ?$$

Algebraic solution

$$[1,2]x = [1,4]$$

$$x = [1,2]$$

because

$$[1,2][1,2] = [1,4]$$

Algebraic solution

$$[1,4]x = [1,4]$$

$$x = [1,1] = 1$$

because

$$[1,4] \cdot 1 = [1,4]$$

Algebraic solution

$$[1,8]x = [1,4]$$

$$x = ?$$

United solution set

$$[1,2]x = [1,4]$$

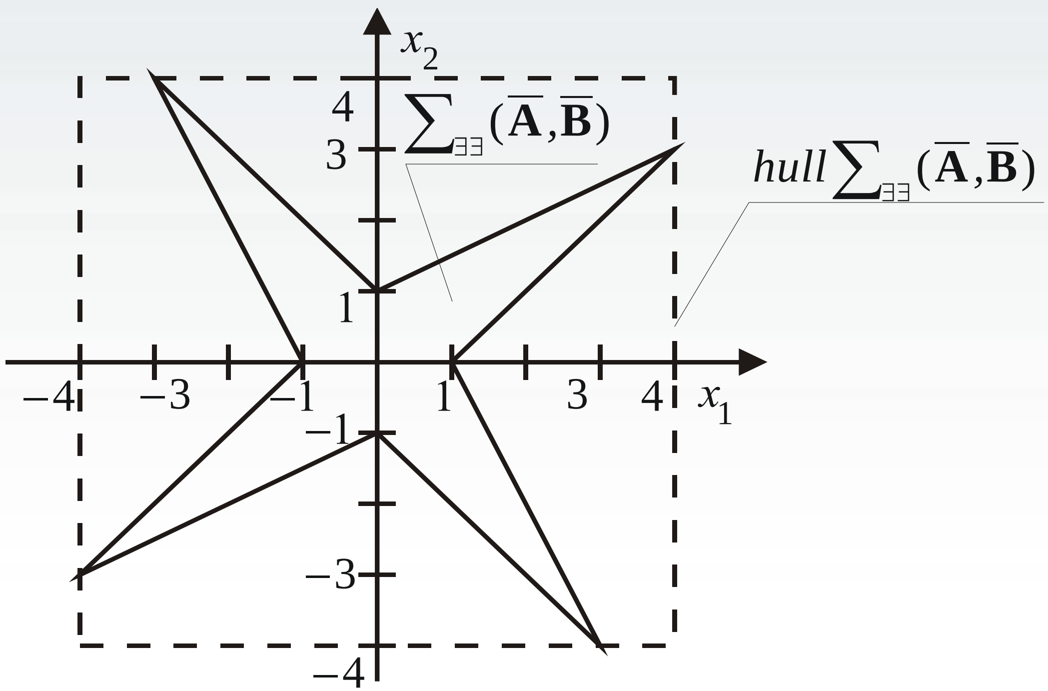
$$\mathbf{x} = \left[\frac{1}{2}, 4\right]$$

because

$$\mathbf{x} = \{x : ax = b, a \in [1,2], b \in [1,4]\}$$

United solution set

$$\begin{bmatrix} [1,2] & [2,4] \\ [2,4] & [1,2] \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} [-1,1] \\ [1,2] \end{bmatrix}$$



Different solution sets

United solution set

$$\sum_{\exists\exists}(\mathbf{A},\mathbf{b}) = \{x : \exists A \in \mathbf{A}, \exists b \in \mathbf{b}, Ax = b\}$$

Tolerable solution set

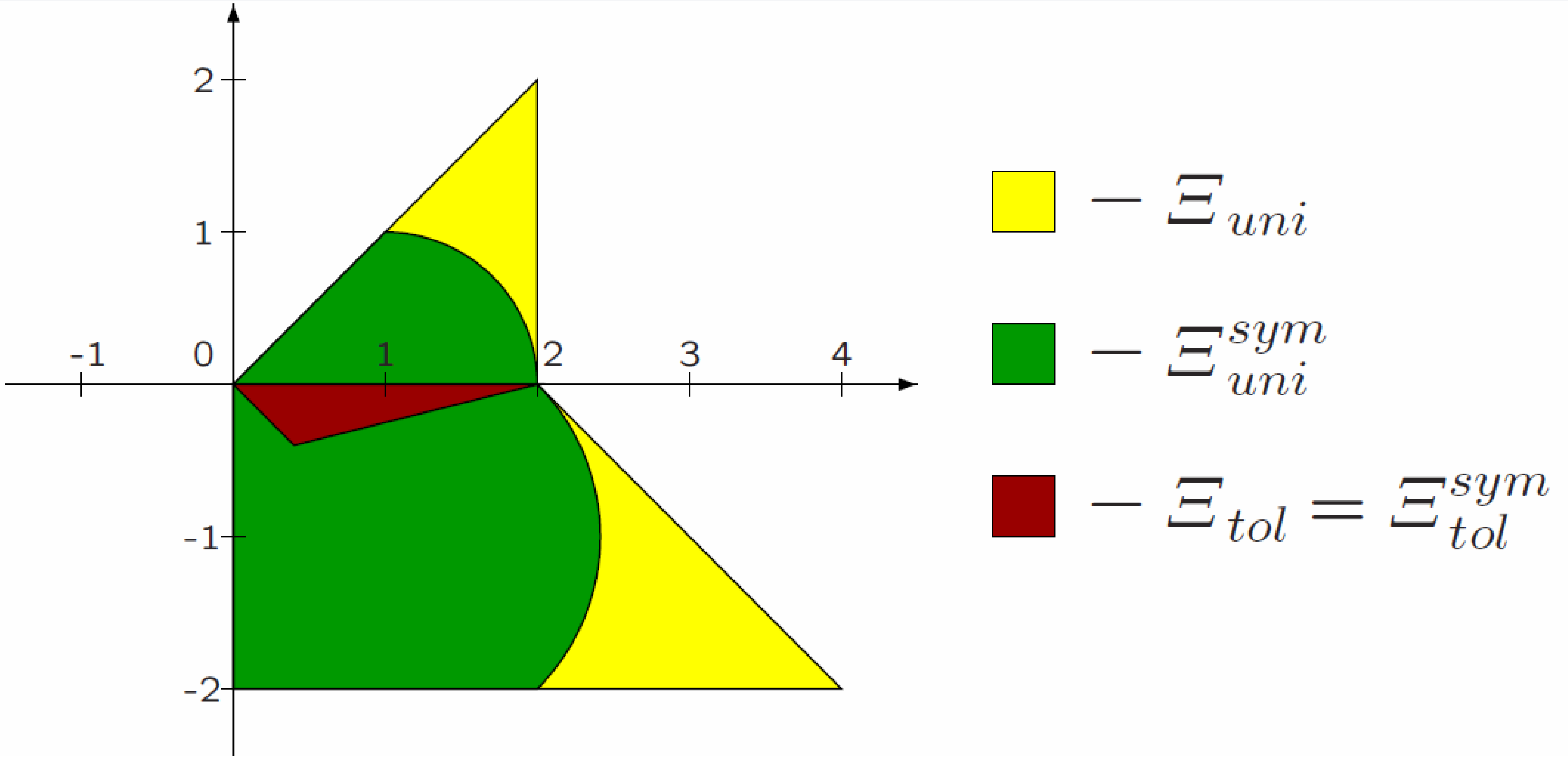
$$\sum_{\forall\exists}(\mathbf{A},\mathbf{b}) = \{x : \forall A \in \mathbf{A}, \exists b \in \mathbf{b}, Ax = b\}$$

Controllable solution set

$$\sum_{\exists\forall}(\mathbf{A},\mathbf{b}) = \{x : \exists A \in \mathbf{A}, \forall b \in \mathbf{b}, Ax = b\}$$

Other solution sets

$$A = \begin{pmatrix} 1 & [0,1] \\ [0,1] & [-4,-1] \end{pmatrix}, \quad b = \begin{pmatrix} [0,2] \\ [0,2] \end{pmatrix},$$



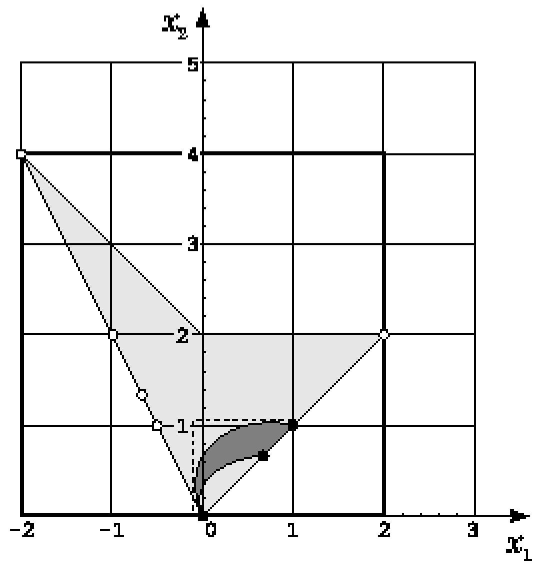
Parametric solution sets

Thus, the *Parametric Solution Set* (PSS) is defined as:

$$\Sigma(\mathbf{A}(\mathbf{p}), \mathbf{b}(\mathbf{p})) = \{\, \tilde{\mathbf{x}} \in \mathbb{R}^n \mid (\exists\, \tilde{\mathbf{p}} \in \mathbf{p})\ \mathbf{A}(\tilde{\mathbf{p}})\tilde{\mathbf{x}} = \mathbf{b}(\tilde{\mathbf{p}}) \,\}.$$

Contrary to the normal case, *Parametric Boundary Solution Set* does not in general contain extremal points of PSS, so at most:

$$\text{hull}\,\Sigma(\mathbf{A}(\partial\mathbf{p}), \mathbf{b}(\partial\mathbf{p})) \subseteq \text{hull}\,\Sigma(\mathbf{A}(\mathbf{p}), \mathbf{b}(\mathbf{p})).$$



- $\Sigma(\mathbf{A}, \mathbf{b})$
- $\Sigma(\partial\mathbf{A}, \partial\mathbf{b})$
- hull $\Sigma(\mathbf{A}, \mathbf{b})$
- $\Sigma(\mathbf{A}(\mathbf{p}), \mathbf{b}(\mathbf{p}))$
- $\Sigma(\mathbf{A}(\partial\mathbf{p}), \mathbf{b}(\partial\mathbf{p}))$
- hull $\Sigma(\mathbf{A}(\mathbf{p}), \mathbf{b}(\mathbf{p}))$

$$\mathbf{A} = \begin{bmatrix} [0, 1] & [1, 2] \\ -2 & [-1, 2] \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} [0, 2] \\ 0 \end{bmatrix},$$

$$\mathbf{A}(\mathbf{p}) = \begin{bmatrix} p_1 & 1+p_2 \\ -2 & 3p_1-1 \end{bmatrix} \quad \mathbf{b}(\mathbf{p}) = \begin{bmatrix} 2p_1 \\ 0 \end{bmatrix}, \quad p_1, p_2 \in [0, 1]$$

http://www.ippt.gov.pl/~kros/pccmm99/10PSS.html

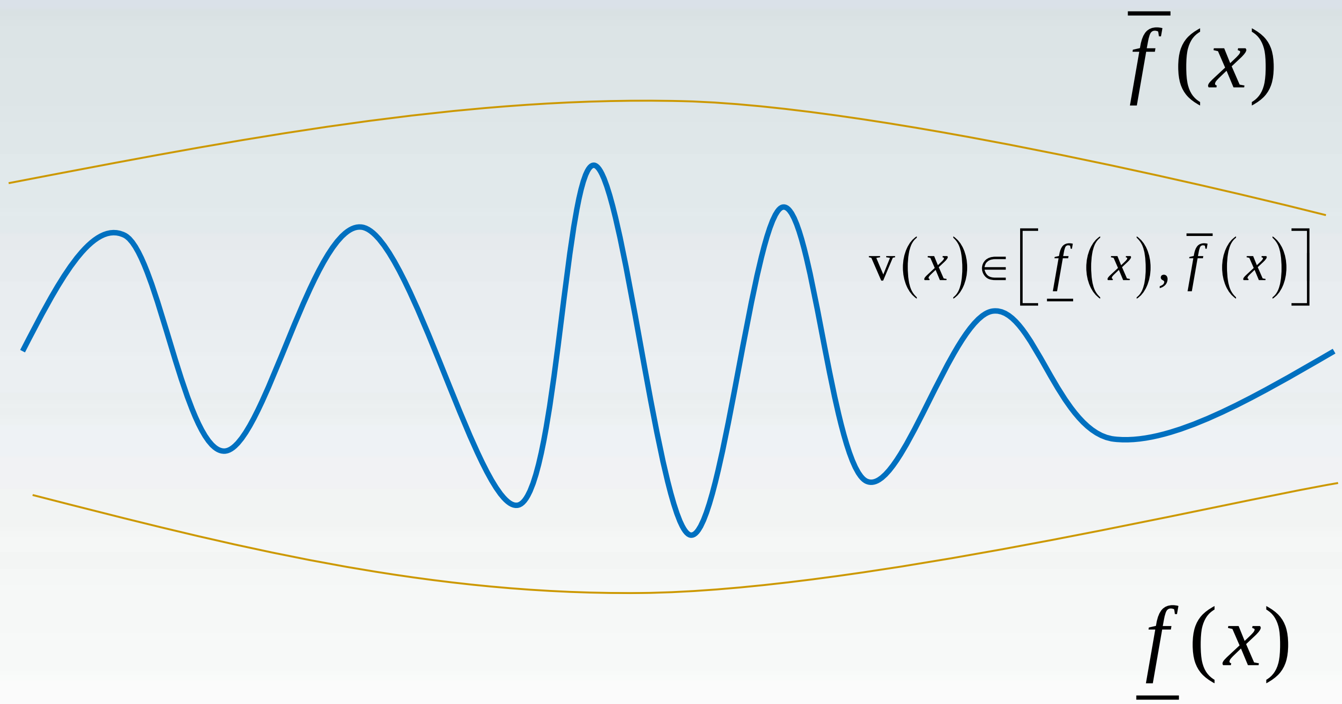
Derivative

$$f(x) = \left[\underline{f}(x), \overline{f}(x) \right]$$

$$f'(x) = \left[\min\left\{ \underline{f}'(x), \overline{f}'(x) \right\}, \max\left\{ \underline{f}'(x), \overline{f}'(x) \right\} \right]$$

What is the definition of the solution of differential equation?

Differentiation of the interval function



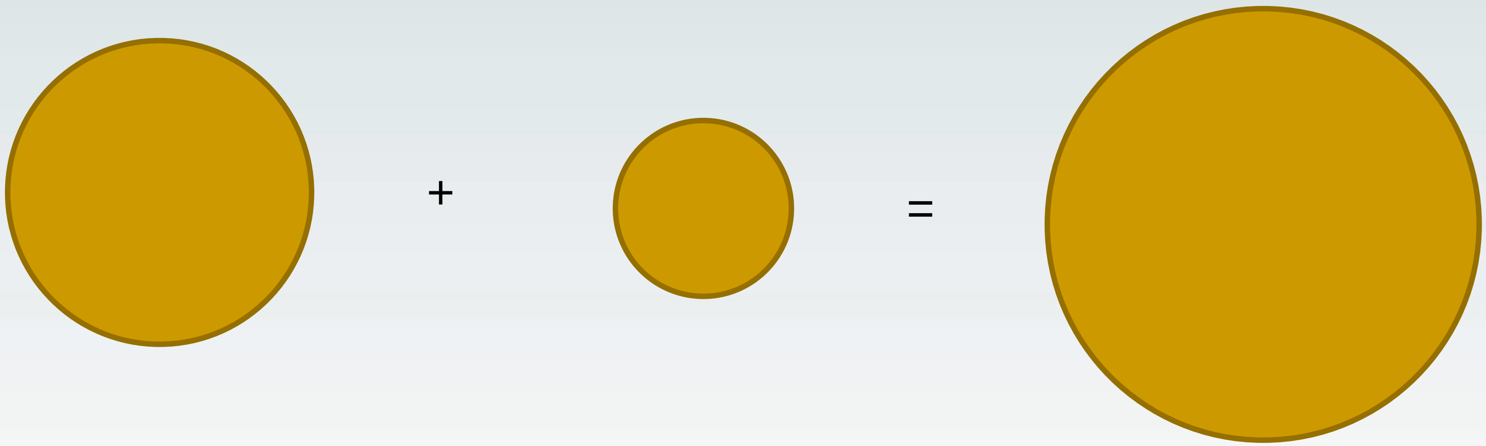
$$\mathrm{v}'(x) \notin \left[\min\left\{ \underline{f}'(x), \overline{f}'(x) \right\}, \max\left\{ \underline{f}'(x), \overline{f}'(x) \right\} \right]$$

What is the definition of the solution of differential equation?

Dubois D., Prade H., 1987, On Several Definition of the Differentiation of Fuzzy Mapping, Fuzzy Sets and Systems, Vol.24, pp.117-120

How about integral equations?

Circular complex arithmetic



$$\mathbf{z}_1 + \mathbf{z}_2 = \langle c_1, \rho_1 \rangle + \langle c_2, \rho_2 \rangle = \langle c_1 + c_2, \rho_1 + \rho_2 \rangle$$

$$\mathbf{z} = \{ z : |z - c| \le \rho \} = \langle c, \rho \rangle$$

Other problems

- Modal interval arithmetic
- Affine arithmetic
- Constrain interval arithmetic
- Ellipsoidal arithmetic
- Convex models (equations with the ellipsoidal parameters)
- General set valued arithmetic
- Fuzzy relational equations
- …. Etc.

What can we do with that?

?

Modeling of uncertainty

- Parametric solution set

Set valued solution

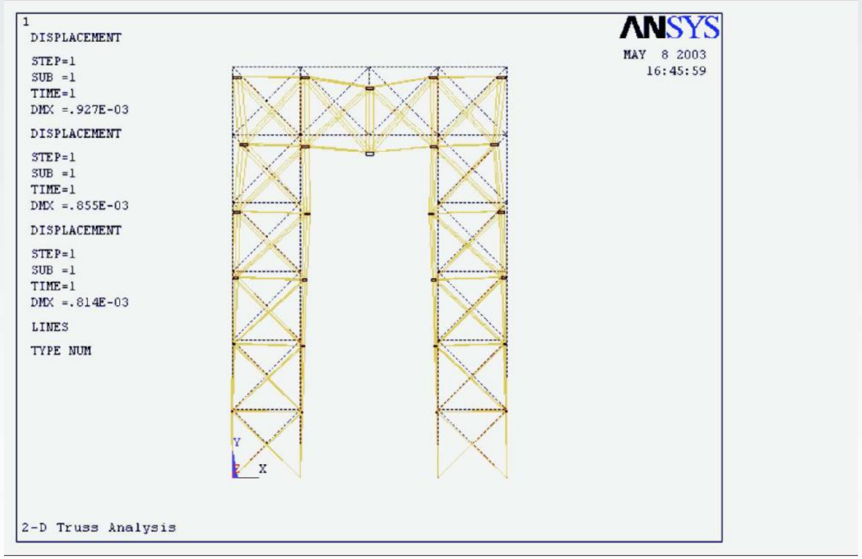
$$\tilde{u}\left(x\right)=\left\{u(x,p):A(x,p,u)=b(x,p),\, p\in\mathbf{p}\right\}$$

Interval solution

$$\mathbf{u}(x)=\Diamond\tilde{u}\left(x\right)=\Diamond\left\{u(x,p):A(x,p,u)=b(x,p),\, p\in\mathbf{p}\right\}$$

Examples of applications

Uncertain truss structures in ANSYS



No	Error %	No	Error %	No	Error %	No	Error %
1	107.057 %	21	42.4163 %	41	109.399 %	61	31.4828 %
2	78.991 %	22	34.0332 %	42	20.0367 %	62	95.903 %
3	38.2972 %	23	9.30111 %	43	109.399 %	63	48.1069 %
4	52.0345 %	24	100.427 %	44	0.833207 %	64	68.1526 %
5	22.7834 %	25	0.833207 %	45	116.216 %	65	22.7834 %
6	68.1526 %	26	116.216 %	46	100.427 %	66	52.0345 %
7	95.903 %	27	20.0367 %	47	34.0332 %	67	78.991 %
8	48.1069 %	28	116.216 %	48	9.30111 %	68	38.2972 %
9	31.4828 %	29	48.1793 %	49	42.4163 %	69	107.057 %
10	22.6152 %	30	116.216 %	50	15.3219 %		
11	38.0037 %	31	8.87714 %	51	33.6609 %		
12	91.4489 %	32	19.1787 %	52	119.633 %		
13	45.9704 %	33	35.7319 %	53	47.414 %		
14	11.913 %	34	18.7494 %	54	9.99375 %		
15	24.3824 %	35	6.38613 %	55	24.3824 %		
16	9.99375 %	36	18.7494 %	56	11.913 %		
17	119.633 %	37	19.1787 %	57	91.4489 %		
18	56.7357 %	38	35.7319 %	58	45.9704 %		
19	33.6609 %	39	8.87714 %	59	38.0037 %		
20	15.3219 %	40	48.1793 %	60	22.6152 %		

Taylor expansion method

$$u_{i,L}^{-}\left(\hat{\mathrm{h}}\right) = u_i\left(\mathrm{h}_0\right) - \sum_{\alpha=1}^{m} \left|\frac{\partial u_i\left(\mathrm{h}_0\right)}{\partial h_\alpha}\right| \left(h_\alpha^{+} - h_{\alpha,0}\right),$$

$$u_{i,L}^{+}\left(\hat{\mathrm{h}}\right) = u_i\left(\mathrm{h}_0\right) + \sum_{\alpha=1}^{m} \left|\frac{\partial u_i\left(\mathrm{h}_0\right)}{\partial h_\alpha}\right| \left(h_\alpha^{+} - h_{\alpha,0}\right).$$

Pownuk, A.,
Efficient Method of Solution of Large Scale Engineering Problems with Interval Parameters Based on Sensitivity Analysis ,
Proceeding of NSF workshop on Reliable Engineering Computing ,
September 15-17, 2004, Savannah, Georgia, USA, pp. 305-316

Sensitivity analysis method

$$y = f(x), \quad x \in [x^-, x^+].$$

If $\dfrac{\partial f(x_0)}{\partial x} > 0$, then $y^- = y(x^-), y^+ = y(x^+)$

If $\dfrac{\partial f(x_0)}{\partial x} < 0$, then $y^- = y(x^+), y^+ = y(x^-)$

$$y = x^2, x \in [1,3]$$

$$\frac{dy(x)}{dx} = 2 \cdot x, \ \frac{dy(2)}{dx} = 2 \cdot 2 = 4, \ y^- = y(x^-) = 1, \ y^+ = y(x^+) = 9$$

$$\hat{y} = [1,9]$$

Sensitivity analysis

- Pownuk A., Numerical solutions of fuzzy partial differential equation and its application in computational mechanics, Fuzzy Partial Differential Equations and Relational Equations: Reservoir Characterization and Modeling (M. Nikravesh, L. Zadeh and V. Korotkikh, eds.), Studies in Fuzziness and Soft Computing, Physica-Verlag, 2004, pp. 308-347

$$A(p)u = b(p)$$

$$u(p) = \left[A(p)\right]^{-1} b(p)$$

$$A(p)\frac{\partial u}{\partial p_k} = \frac{\partial b(p)}{\partial p_k} - \frac{\partial A(p)}{\partial p_k}u$$

Applications

- http://andrzej.pownuk.com/interval_web_applications.htm

Monotonicity tests

$$\frac{\partial u}{\partial p_i} \approx \frac{\partial u}{\partial p_i} + \sum_j \frac{\partial^2 u}{\partial p_i \partial p_j}\left(p_j - p_{j0}\right)$$

$$\left(\frac{\partial u}{\partial p_i}\right)^- \approx \frac{\partial u}{\partial p_i} - \sum_j \left|\frac{\partial^2 u}{\partial p_i \partial p_j}\right| \Delta p_j$$

$$\left(\frac{\partial u}{\partial p_i}\right)^+ \approx \frac{\partial u}{\partial p_i} + \sum_j \left|\frac{\partial^2 u}{\partial p_i \partial p_j}\right| \Delta p_j$$

Higher order monotonicity tests

$$\frac{\partial^2 u}{\partial p_i \partial p_j} \approx \frac{\partial^2 u}{\partial p_i \partial p_j} + \sum_j \frac{\partial^3 u}{\partial p_i \partial p_j \partial p_k} \left(p_k - p_{k0} \right)$$

$$\left(\frac{\partial u}{\partial p_i \partial p_j} \right)^- \approx \frac{\partial u}{\partial p_i \partial p_j} - \sum_j \left| \frac{\partial^2 u}{\partial p_i \partial p_j \partial p_k} \right| \Delta p_k$$

$$\left(\frac{\partial u}{\partial p_i \partial p_j} \right)^+ \approx \frac{\partial u}{\partial p_i \partial p_j} + \sum_j \left| \frac{\partial^2 u}{\partial p_i \partial p_j \partial p_k} \right| \Delta p_k$$

Example

- $f(x) = 432 - 972 x + 900 x^2 - 439 x^3 + 119 x^4 - 17 x^5 + x^6$
- $df(x) = -972 + 1800 x - 1317 x^2 + 476 x^3 - 85 x^4 + 6 x^5$
- $d2f(x) = 1800 - 2634 x + 1428 x^2 - 340 x^3 + 30 x^4$
- $d3f(x) = -2634 + 2856 x - 1020 x^2 + 120 x^3$
- $d4f(x) = 2856 - 2040 x + 360 x^2$
- $d5f(x) = -2040 + 720 x$
- $d6f(x) = 720$

Example

- Traditional approach [x]=[0,1]
- f([x]) = [-996, 2880]
- df([x]) = [-4656, 1310]
- d2f([x]) = [-1174, 6232]
- d3f([x]) = [-6630, 342]
- d4f([x]) = [816, 5256]
- d5f([x]) = [-2760, -1320]
- d6f([x]) = 720

Example

- New approach
- {f(xl),f(xu)} = {2880, 24}
- {df(xl),df(xu)} = {-4656, -92}
- {d2f(xl),d2f(xu)} = {6232, 284}
- {d3f(xl),d3f(xu)} = {-6630, -678}
- {d4f(xl),d4f(xu)} = {5256, 1176}
- {d5f(xl),d5f(xu)} = {-2760, -1320}
- {d6f(xl),d6f(xu)} = {720, 720}

Functions with more than one variable

- $f(x,y)=-27 + 9 x + 18 y - 6 x y - 3 y^2 + x y^2$
- $fx=9 - 6 y + y^2$
- $fy=18 - 6 x - 6 y + 2 x y$
- $fxx=0$
- $fxy=-6 + 2 y$
- $fyy=-6 + 2 x$
- $fxxx=0$
- $fxxy=0$
- $fxyy=2$
- $fyyy=0$

Extreme values of second order derivatives

- Range of fxx=0 i.e. constant sign.
- Extreme values of fxy=-6+2y
- Because fxyx=0 then
- xmin=xl
- xmax=xu
- Because fxyy=2>0 then
- ymin=yl
- ymax=yu
- Range of fxy=[fxy[xmin, ymin], fxy[xmax, ymax]]=
- =[-6, -4].

Extreme values of second order derivatives

- Extreme values of fyy=-6+2x
- Because fyyx=2>0 then
- xmin=xl
- xmax=xu
- Because fyyy=0 then
- ymin=yl
- ymax=yu
- Range of fyy=[fyy[xmin, ymin], fyy[xmax, ymax]]=

=[-6, -4]

Extreme values of first order derivatives

- Range of fx = 9 - 6 y + y^2
- Because fxx=0 then
- xmin=xl
- xmax=xu
- Because fxy=[-6,-4]<0 then
- ymin=yu
- ymax=yl
- Range of fx=[fx[xmin, ymin], fx[xmax, ymax]]={4, 9}

Extreme values of first order derivatives

- Range of fy=18-6 x-6 y+2 x y
- Because fyx=[-6,-4]<0 then
- xmin=xu
- xmax=xl
- Because fyy=[-6,-4]<0 then
- ymin=yu
- ymax=yl
- Range of fy=[fy[xmin, ymin], fy[xmax, ymax]] = =[8, 18]

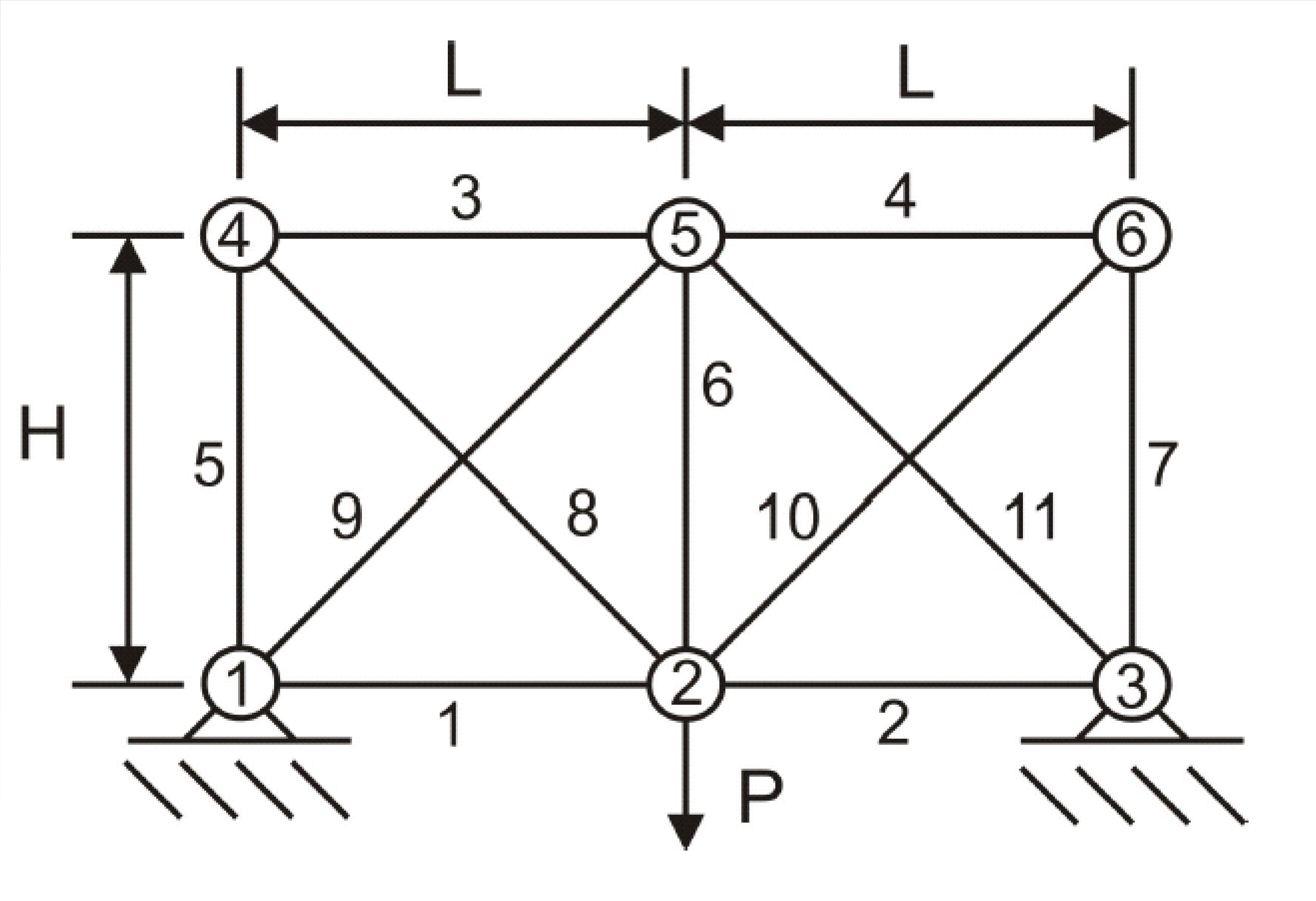
Zero order derivatives (Range of function)

- Because fx=[4,9]>0 then
- xmin=xl
- xmax=xu

- Because fy=[8,18]>0 then
- ymin=yl
- ymax=yu

- Range of function
- f=[f[xmin, ymin], f[xmax, ymax]]=[-27, -8]

Truss structures



Analytical solution

```
{{-(10 000 (-(0.00353553 E4 + 0.01 E6 + 0.00353553 E8)
(0.00353553 E7 (-0.00353553 E7 (0.01 E11 (-1.×10^-6 E10 E11 E3 - 3.53553×10^-7 E10 E11 E5 - 3.53553×10^-7 E11 E3 E5 + 0. E11 E5^2) + (0.01 E11 + 0.00353553 E7) (-0.01 E10^2 (0.0001 E3 + 0.0000353553 E5) +
(0.0001 E10 E3 + 0.0000353553 E10 E5 + 0.0000353553 E3 E5 + 0. E5^2) (0.01 E10 + 0.01 E11 + 0.00353553 E4 + 0.00353553 E8))) + 0.00353553 E7 (E10 E11 (3.53553×10^-9 E3 E5 + 0. E5^2) +
0.00353553 E7 (-0.01 E10^2 (0.0001 E3 + 0.0000353553 E5) + (0.0001 E10 E3 + 0.0000353553 E10 E5 + 0.0000353553 E3 E5 + 0. E5^2) (0.01 E10 + 0.01 E11 + 0.00353553 E4 + 0.00353553 E8)))) -
0.00353553 E7 (-0.00353553 E7 (E10 E11 (-3.53553×10^-9 E3 E5 + 0. E5^2) + 0.00353553 E7 (-0.01 E10^2 (0.0001 E3 + 0.0000353553 E5) + (0.0001 E10 E3 + 0.0000353553 E10 E5 + 0.0000353553 E3 E5 + 0. E5^2)
(0.01 E10 + 0.01 E11 + 0.00353553 E4 + 0.00353553 E8))) + 0.00353553 E7 (-0.01 E10^2 (-3.53553×10^-7 E3 E5 + 0. E5^2 + 3.53553×10^-7 E3 E7 + 1.25×10^-7 E5 E7) +
(0. E3 E5^2 + 0. E5^3 + 1.25×10^-7 E3 E5 E7 + 0. E5^2 E7 + E10 (-3.53553×10^-7 E3 E5 + 0. E5^2 + 3.53553×10^-7 E3 E7 + 1.25×10^-7 E5 E7)) (0.01 E10 + 0.01 E11 + 0.00353553 E4 + 0.00353553 E8))) +
(0.01 E11 (E11 (0. E3 E5^2 + 0. E5^3 - 1.25×10^-9 E3 E5 E7 + 0. E5^2 E7) + E10 (1.25×10^-9 E3 E5 E7 + 0. E5^2 E7 + E11 (3.53553×10^-9 E3 E5 + 0. E5^2 - 3.53553×10^-9 E3 E7 - 1.25×10^-9 E5 E7))) -
0.00353553 E7 (E10 E11 (3.53553×10^-9 E3 E5 + 0. E5^2) + 0.00353553 E7 (-0.01 E10^2 (0.0001 E3 + 0.0000353553 E5) + (0.0001 E10 E3 + 0.0000353553 E10 E5 + 0.0000353553 E3 E5 + 0. E5^2)
(0.01 E10 + 0.01 E11 + 0.00353553 E4 + 0.00353553 E8))) + (0.01 E11 + 0.00353553 E7) (-0.01 E10^2 (-3.53553×10^-7 E3 E5 + 0. E5^2 + 3.53553×10^-7 E3 E7 + 1.25×10^-7 E5 E7) + (0. E3 E5^2 + 0. E5^3 +
1.25×10^-7 E3 E5 E7 + 0. E5^2 E7 + E10 (-3.53553×10^-7 E3 E5 + 0. E5^2 + 3.53553×10^-7 E3 E7 + 1.25×10^-7 E5 E7)) (0.01 E10 + 0.01 E11 + 0.00353553 E4 + 0.00353553 E8))) (0.00353553 E7 + 0.01 E9)) +
(0.00353553 E4 - 0.00353553 E8) (0.01 E6 (E11 (E3 E5 (0. E7^2 - 1.25×10^-11 E7 E9) + E5^2 (0. E7^2 + 0. E7 E9)) + E10 (0. E3 E5 E7^2 + 0. E5^2 E7^2 - 1.25×10^-11 E3 E5 E7 E9 + 0. E5^2 E7 E9 +
E11 (0. E5^2 E7 + 0. E5 E7^2 + 0. E5^2 E9 - 1.25×10^-11 E5 E7 E9 + E3 (-1.25×10^-11 E5 E7 + 0. E7^2 - 3.53553×10^-11 E5 E9 - 3.53553×10^-11 E7 E9)))) +
(0.00353553 E4 - 0.00353553 E8) (0. E3 E5^2 E7^2 + 0. E5^3 E7^2 + 0. E3 E5 E7^3 + 0. E5^2 E7^3 + 0. E3 E5^2 E7 E9 + 0. E5^3 E7 E9 + 0. E3 E5 E7^2 E9 + 0. E5^2 E7^2 E9 +
E11 (E5^3 (0. E7 + 0. E9) + E5^2 (0. E7^2 + 0. E7 E9) + E3 (E5^2 (0. E7 + 0. E9) + E5 (0. E7^2 + 1.25×10^-11 E7 E9))) + E10 (0. E5^2 E7^2 + 0. E5 E7^3 + 0. E5^2 E7 E9 + 0. E5 E7^2 E9 +
E3 (0. E5 E7^2 + 0. E7^3 - 1.25×10^-11 E5 E7 E9 + 0. E7^2 E9) + E11 (0. E5^2 E7 + 0. E5 E7^2 + 0. E5^2 E9 + 1.25×10^-11 E5 E7 E9 + E3 (-1.25×10^-11 E5 E7 + 0. E7^2 - 3.53553×10^-11 E5 E9 + 3.53553×10^-11 E7 E9))))))) /
(-0.01 E6 (0.01 E6 (0.00353553 E7 (-0.00353553 E7 (0.01 E11 (-1.×10^-6 E10 E11 E3 - 3.53553×10^-7 E10 E11 E5 - 3.53553×10^-7 E11 E3 E5 + 0. E11 E5^2) + (0.01 E11 + 0.00353553 E7)
(-0.01 E10^2 (0.0001 E3 + 0.0000353553 E5) + (0.0001 E10 E3 + 0.0000353553 E10 E5 + 0.0000353553 E3 E5 + 0. E5^2) (0.01 E10 + 0.01 E11 + 0.00353553 E4 + 0.00353553 E8))) + 0.00353553
E7 (E10 E11 (3.53553×10^-9 E3 E5 + 0. E5^2) + 0.00353553 E7 (-0.01 E10^2 (0.0001 E3 + 0.0000353553 E5) + (0.0001 E10 E3 + 0.0000353553 E10 E5 + 0.0000353553 E3 E5 + 0. E5^2)
(0.01 E10 + 0.01 E11 + 0.00353553 E4 + 0.00353553 E8)))) - 0.00353553 E7 (-0.00353553 E7 (E10 E11 (3.53553×10^-9 E3 E5 + 0. E5^2) +
0.00353553 E7 (-0.01 E10^2 (0.0001 E3 + 0.0000353553 E5) + (0.0001 E10 E3 + 0.0000353553 E10 E5 + 0.0000353553 E3 E5 + 0. E5^2) (0.01 E10 + 0.01 E11 + 0.00353553 E4 + 0.00353553 E8))) +
0.00353553 E7 (-0.01 E10 (E1 E10 (1.×10^-6 E3 + 3.53553×10^-7 E5) + E10 (1.×10^-6 E2 E3 + 3.53553×10^-7 E2 E5 + 3.53553×10^-7 E3 E5 + 0. E5^2 + 3.53553×10^-7 E3 E7 + 1.25×10^-7 E5 E7)) +
(3.53553×10^-7 E2 E3 E5 + 0. E2 E5^2 + 0. E3 E5^2 + 0. E5^3 + E1 (1.×10^-6 E10 E3 + 3.53553×10^-7 E10 E5 + 3.53553×10^-7 E3 E5 + 0. E5^2) + 1.25×10^-7 E3 E5 E7 + 0. E5^2 E7 +
E10 (1.×10^-6 E2 E3 + 3.53553×10^-7 E2 E5 + 3.53553×10^-7 E3 E5 + 0. E5^2 + 3.53553×10^-7 E3 E7 + 1.25×10^-7 E5 E7)) (0.01 E10 + 0.01 E11 + 0.00353553 E4 + 0.00353553 E8))) +
(0.01 E11 (-3.53553×10^-9 E11 E2 E3 E5 + 0. E11 E2 E5^2 + 0. E11 E3 E5^2 + 0. E11 E5^3 + E1 (-1.×10^-8 E10 E11 E3 - 3.53553×10^-9 E10 E11 E5 - 3.53553×10^-9 E11 E3 E5 + 0. E11 E5^2) -
1.25×10^-9 E11 E3 E5 E7 + 0. E11 E5^2 E7 + E10 (-1.25×10^-9 E3 E5 E7 + 0. E5^2 E7 + E11 (-1.×10^-8 E2 E3 - 3.53553×10^-9 E2 E5 - 3.53553×10^-9 E3 E5 + 0. E5^2 - 3.53553×10^-9 E3 E7 - 1.25×10^-9 E5 E7))) -
0.00353553 E7 (E10 E11 (3.53553×10^-9 E3 E5 + 0. E5^2) + 0.00353553 E7 (-0.01 E10^2 (0.0001 E3 + 0.0000353553 E5) + (0.0001 E10 E3 + 0.0000353553 E10 E5 + 0.0000353553 E3 E5 + 0. E5^2)
(0.01 E10 + 0.01 E11 + 0.00353553 E4 + 0.00353553 E8))) + (0.01 E11 + 0.00353553 E7)
(-0.01 E10 (E1 E10 (1.×10^-6 E3 + 3.53553×10^-7 E5) + E10 (1.×10^-6 E2 E3 + 3.53553×10^-7 E2 E5 + 3.53553×10^-7 E3 E5 + 0. E5^2 + 3.53553×10^-7 E3 E7 + 1.25×10^-7 E5 E7)) +
(3.53553×10^-7 E2 E3 E5 + 0. E2 E5^2 + 0. E3 E5^2 + 0. E5^3 + E1 (1.×10^-6 E10 E3 + 3.53553×10^-7 E10 E5 + 3.53553×10^-7 E3 E5 + 0. E5^2) + 1.25×10^-7 E3 E5 E7 + 0. E5^2 E7 +
E10 (1.×10^-6 E2 E3 + 3.53553×10^-7 E2 E5 + 3.53553×10^-7 E3 E5 + 0. E5^2 + 3.53553×10^-7 E3 E7 + 1.25×10^-7 E5 E7)) (0.01 E10 + 0.01 E11 + 0.00353553 E4 + 0.00353553 E8))) (0.00353553 E7 + 0.01 E9)) +
(0.00353553 E4 - 0.00353553 E8) (0. E11 E2 E3 E5 E7^2 + 0. E11 E2 E5^2 E7^2 + 0. E11 E3 E5^2 E7^2 + 0. E11 E5^3 E7^2 + 0. E11 E3 E5 E7^3 + 0. E11 E5^2 E7^3 - 1.25×10^-13 E11 E2 E3 E5 E7 E9 +
0. E11 E2 E5^2 E7 E9 + 0. E11 E3 E5^2 E7 E9 + 0. E11 E5^3 E7 E9 + 0. E11 E3 E5 E7^2 E9 + 0. E11 E5^2 E7^2 E9 + E1 (E11 (E3 E5 (0. E7^2 - 1.25×10^-13 E7 E9) + E5^2 (0. E7^2 + 0. E7 E9)) + E10
(0. E3 E5 E7^2 + 0. E5^2 E7^2 + 1.25×10^-13 E3 E5 E7 E9 + 0. E5^2 E7 E9 + E11 (0. E5^2 E7 + 0. E5 E7^2 + 0. E5^2 E9 - 1.25×10^-13 E5 E7 E9 + E3 (1.25×10^-13 E5 E7 + 0. E7^2 + 3.53553×10^-13 E5 E9 - 3.53553×10^-13 E7 E9)))) +
E10 (0. E2 E3 E5 E7^2 + 0. E2 E5^2 E7^2 + 0. E3 E5^2 E7^2 + 0. E5^3 E7^2 + 0. E3 E5 E7^3 + 0. E5^2 E7^3 + 1.25×10^-13 E2 E3 E5 E7 E9 + 0. E2 E5^2 E7 E9 + 0. E3 E5^2 E7 E9 + 0. E5^3 E7 E9 + 0. E3 E5 E7^2
E9 + 0. E5^2 E7^2 E9 + E11 (0. E5^3 E7 + 0. E5^2 E7^2 + 0. E5 E7^3 + 0. E5^3 E9 + 0. E5^2 E7 E9 + 0. E5 E7^2 E9 + E3 (0. E5^2 E7 + 0. E5 E7^2 + 0. E7^3 + 0. E5^2 E9 + 0. E5 E7 E9 + 0. E7^2 E9) +
```

5% uncertainty

	Combinatoric		Gradient free	
DOF	$\underline{u}$	$\overline{u}$	$\underline{u}$	$\overline{u}$
3	0,0,1,0,1,0,0,1,0,0,1,0	$\boxed{0}$,$\boxed{0}$,0,1,0,$\boxed{0}$,1,0,1,1,0,$\boxed{0}$	0,0,1,0,1,$\boxed{1}$,0,1,0,0,1,$\boxed{1}$	$\boxed{1}$,$\boxed{1}$,0,1,0,0,1,0,1,1,0,0
4	0,0,0,0,0,0,0,0,0,0,0,0	1,1,1,1,1,1,1,1,1,1,1,1	0,0,0,0,0,0,0,0,0,0,0,0	1,1,1,1,1,1,1,1,1,1,1,1
7	0,0,1,1,0,1,1,0,0,1,1,1	$\boxed{0}$,$\boxed{0}$,0,0,1,0,0,1,1,0,0,0	0,0,1,1,0,1,1,0,0,1,1,1	$\boxed{1}$,$\boxed{1}$,0,0,1,0,0,1,1,0,0,0
8	0,0,1,1,0,0,1,1,0,1,0,0	$\boxed{0}$,$\boxed{0}$,0,0,1,1,0,0,1,0,1,1	0,0,1,1,0,0,1,1,0,1,0,0	$\boxed{1}$,$\boxed{1}$,0,0,1,1,0,0,1,0,1,1
9	1,1,0,1,0,1,1,0,0,1,1,0	$\boxed{1}$,$\boxed{1}$,1,0,1,$\boxed{1}$,0,1,1,0,0,$\boxed{0}$	$\boxed{0}$,$\boxed{0}$,0,1,0,1,1,0,0,1,1,$\boxed{1}$	1,1,1,0,1,$\boxed{0}$,0,1,1,0,0,0
10	1,0,0,0,0,1,0,0,0,0,0,0	$\boxed{1}$,1,1,1,1,0,1,1,1,1,1,1	$\boxed{0}$,0,0,0,0,1,0,0,0,0,0,0	1,1,1,1,1,0,1,1,1,1,1,1
11	0,0,0,0,0,0,1,0,0,1,1,0	$\boxed{0}$,$\boxed{0}$,1,1,1,1,0,1,1,0,0,1	0,0,0,0,0,0,1,0,0,1,1,0	$\boxed{1}$,$\boxed{1}$,1,1,1,1,0,1,1,0,0,1
12	0,0,1,1,1,0,0,1,0,1,0,0	$\boxed{0}$,$\boxed{0}$,0,0,0,1,1,0,1,0,1,1	0,0,1,1,1,0,0,1,0,1,0,0	$\boxed{1}$,$\boxed{1}$,0,0,0,1,1,0,1,0,1,1

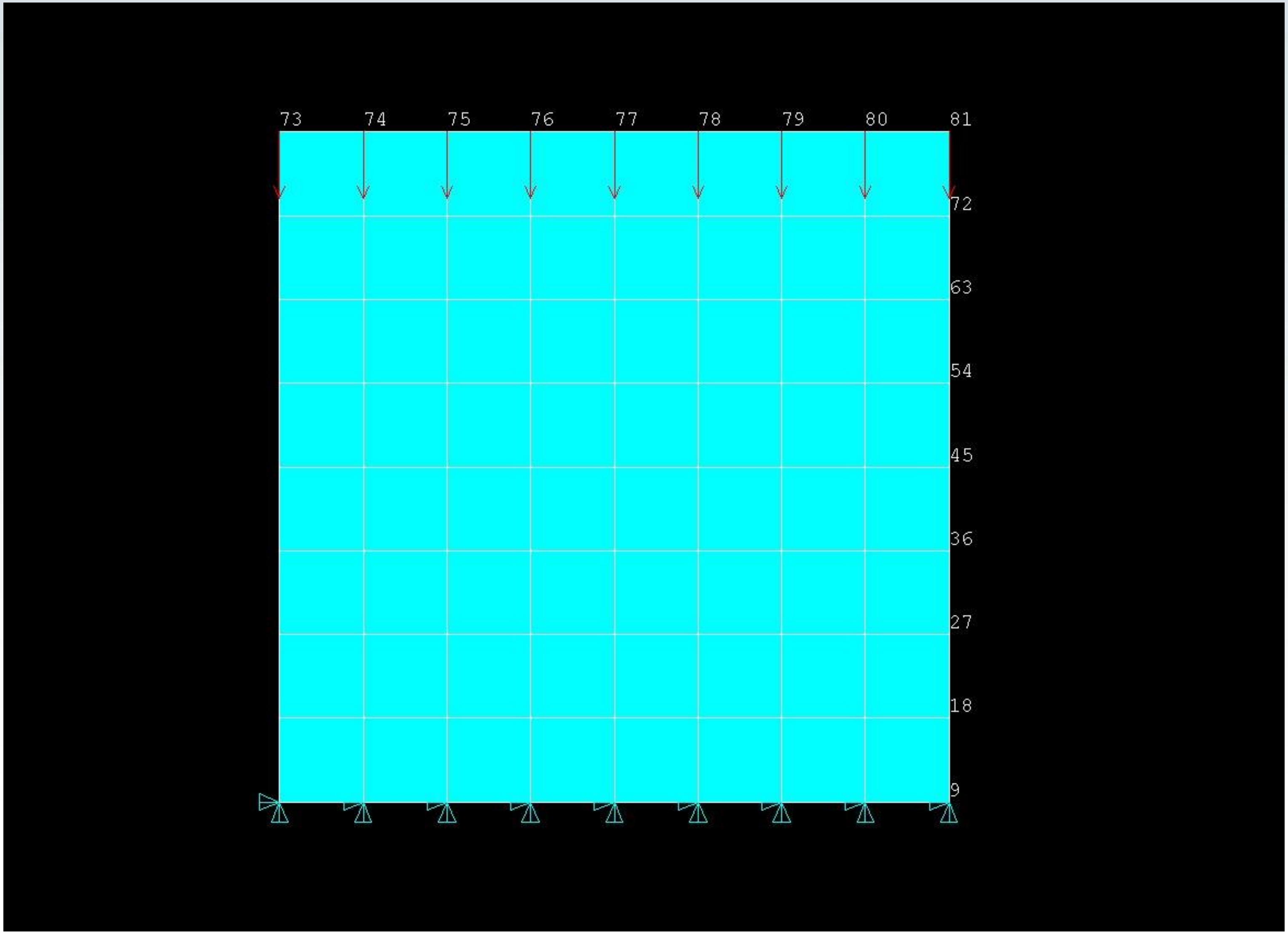
5% uncertainty

	Combinatoric		Gradient free		Error	
DOF	$\underline{u}$	$\overline{u}$	$\underline{u}$	$\overline{u}$	$\underline{u}$ %	$\overline{u}$ %
3	-1.538050E-04	1.538050E-04	-1.388490E-04	1.420150E-04	9.724001E+00	7.665551E+00
4	-1.652290E-02	-1.352560E-02	-1.652290E-02	-1.352560E-02	0.000000E+00	0.000000E+00
7	2.440140E-03	3.537200E-03	2.440140E-03	3.536200E-03	0.000000E+00	2.827095E-02
8	-8.664030E-04	-6.318360E-04	-8.664030E-04	-6.318520E-04	0.000000E+00	2.532303E-03
9	-3.510260E-04	3.510260E-04	-3.172980E-04	3.506350E-04	9.608405E+00	1.113878E-01
10	-1.409620E-02	-1.140120E-02	-1.409620E-02	-1.140120E-02	0.000000E+00	0.000000E+00
11	-3.537200E-03	-2.440140E-03	-3.537200E-03	-2.440960E-03	0.000000E+00	3.360463E-02
12	-8.664030E-04	-6.318360E-04	-8.664030E-04	-6.318520E-04	0.000000E+00	2.532303E-03

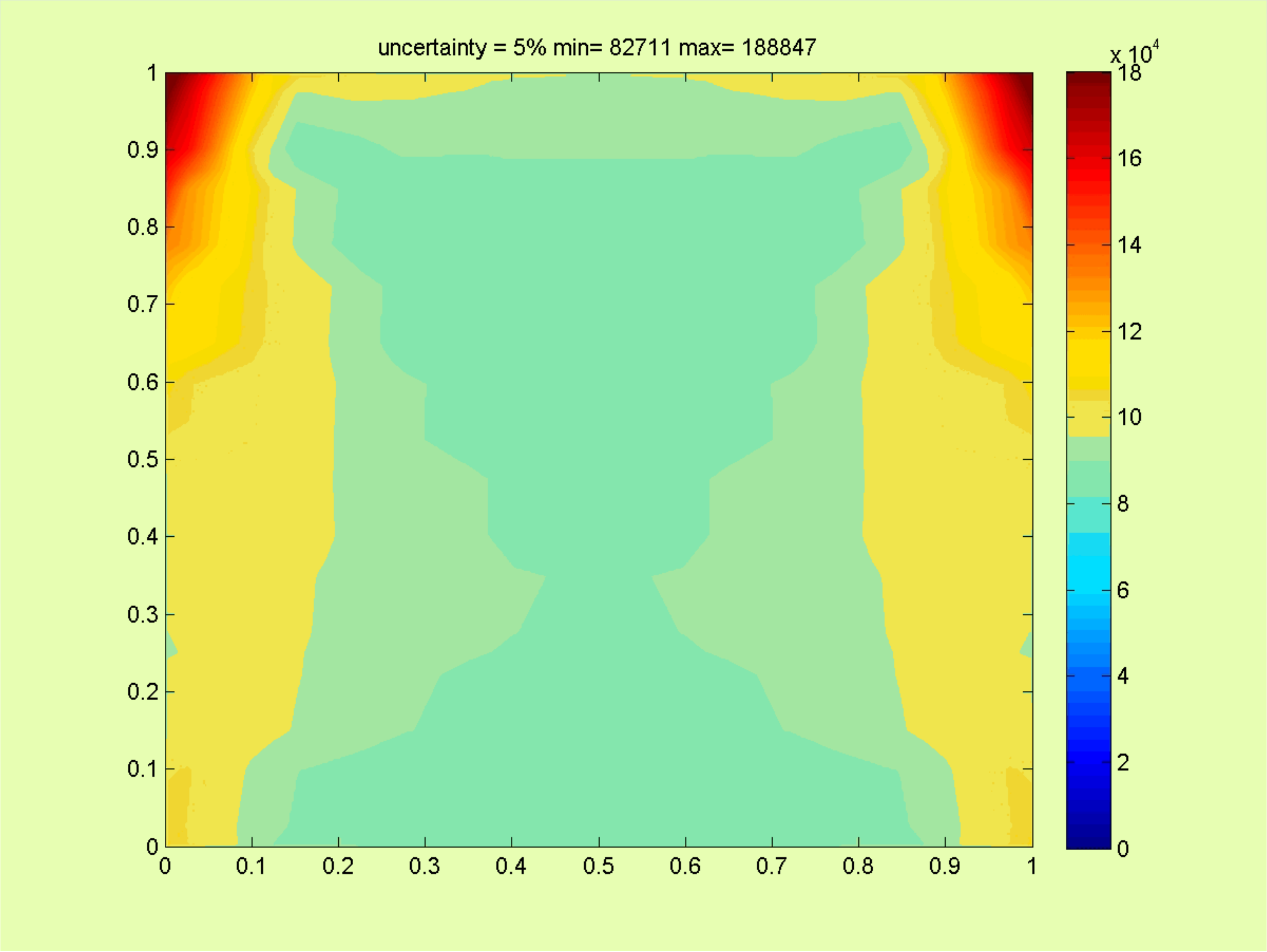
Second order monotonicity test

Derivative	Min	Max	Sign
$\frac{\partial u_5}{\partial E_1}$	-3.26934E-16	2.508830E-16	?
$\frac{\partial u_5}{\partial E_2}$	-3.26934E-16	2.50883E-16	?
$\frac{\partial u_5}{\partial E_3}$	-1.92976E-15	-1.24505E-15	$-$
$\frac{\partial u_5}{\partial E_4}$	1.24505E-15	1.92976E-15	$+$
$\frac{\partial u_5}{\partial E_5}$	-4.97695E-16	-3.01577E-16	$-$
$\frac{\partial u_5}{\partial E_6}$	-1.15435E-16	8.06621E-17	?
$\frac{\partial u_5}{\partial E_7}$	3.01578E-16	4.97696E-16	$+$
$\frac{\partial u_5}{\partial E_8}$	-2.6533E-15	-1.76977E-15	$-$
$\frac{\partial u_5}{\partial E_9}$	2.53431E-15	3.60947E-15	$+$
$\frac{\partial u_5}{\partial E_{10}}$	1.76977E-15	2.6533E-15	$+$
$\frac{\partial u_5}{\partial E_{11}}$	-3.61005E-15	-2.53463E-15	$-$
$\frac{\partial u_5}{\partial E_P}$	-9.13142E-09	9.93587E-09	?

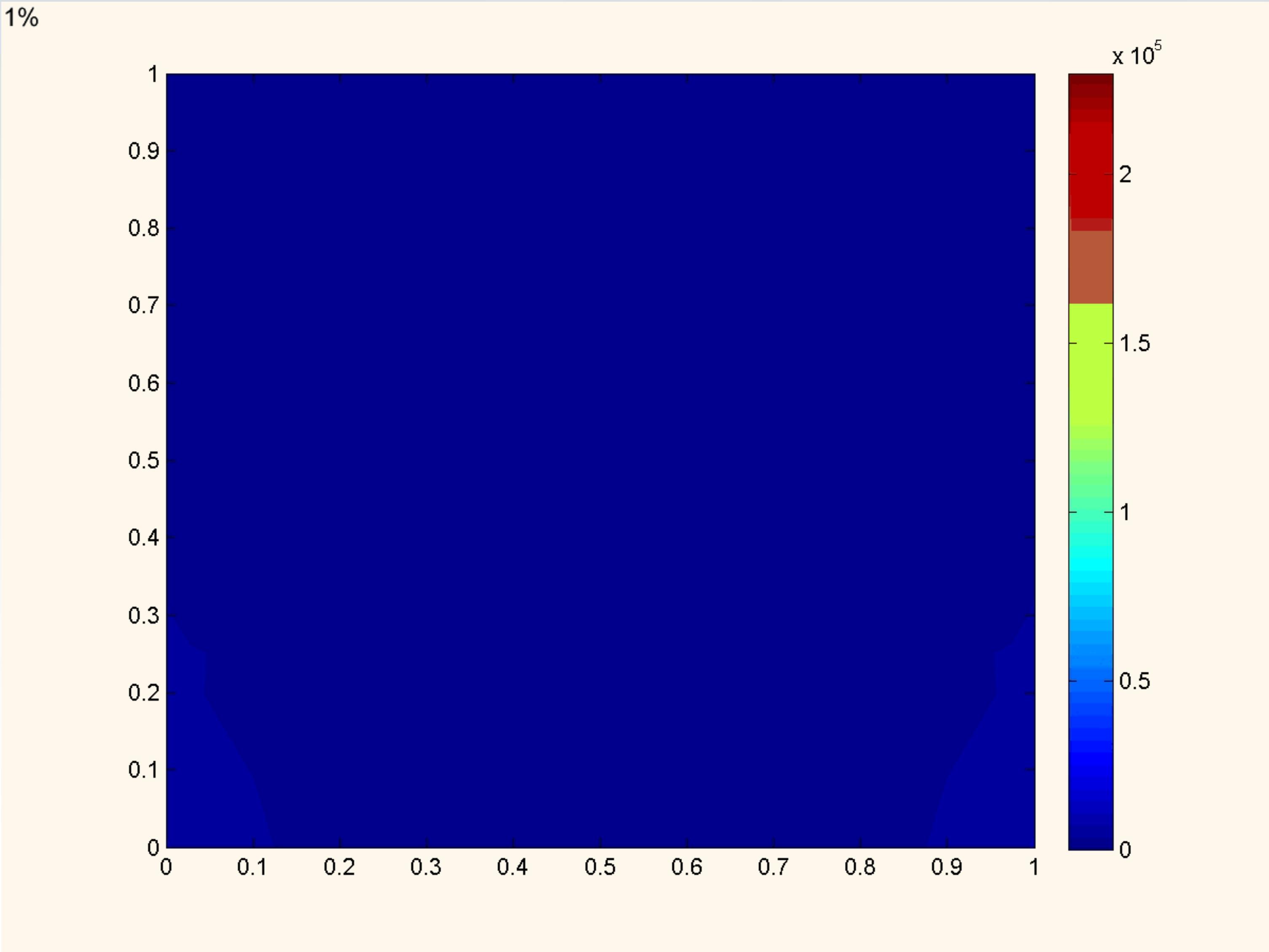
2D example



2D example


uncertainty = 5% min= 82711 max= 188847
x 10^4
18
16
14
12
10
8
6
4
2
0
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

2D Example


1%
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
x 10^5
2
1.5
1
0.5
0

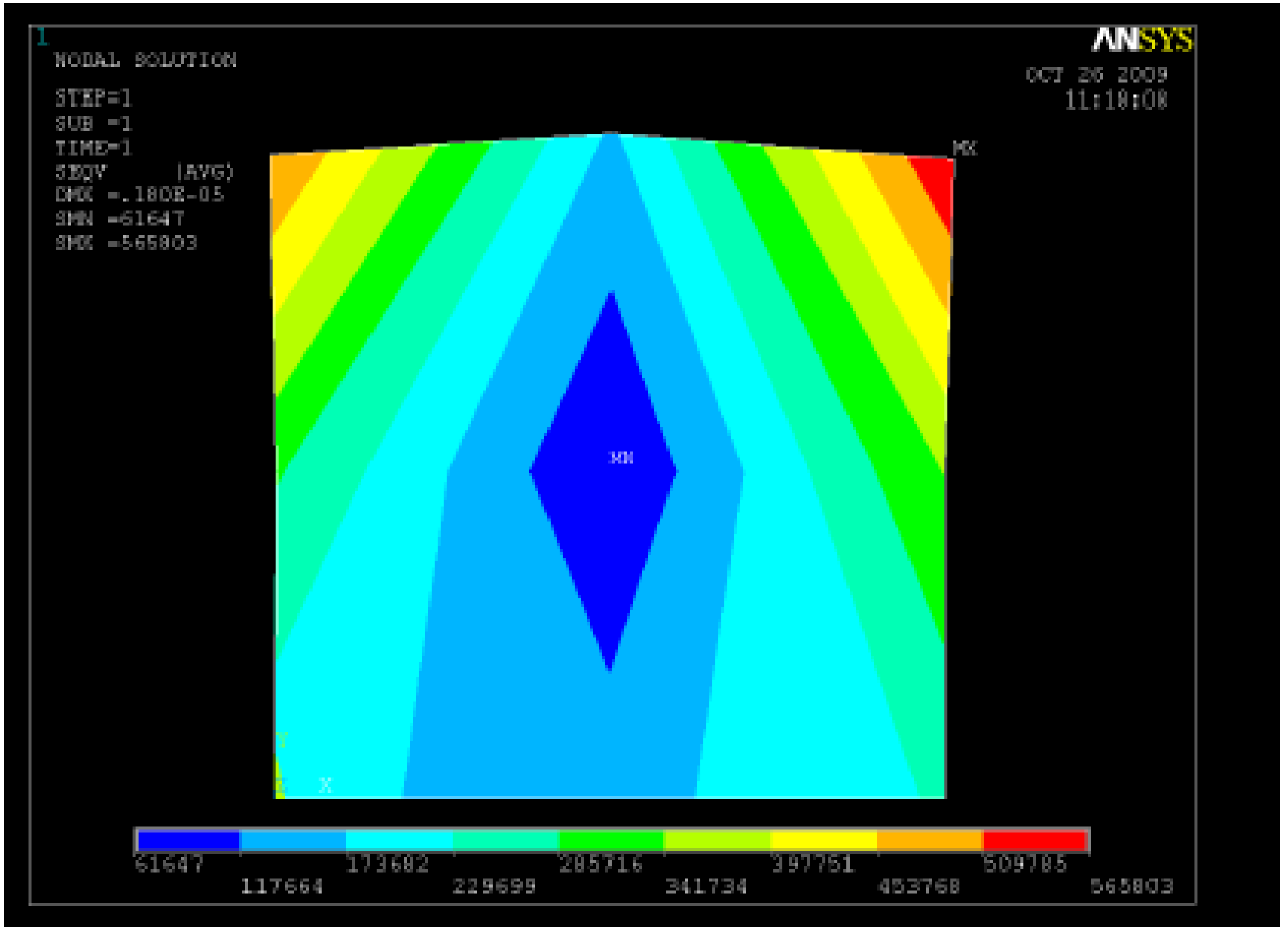
2D example

	Combinatoric	Combinatoric	Gradient free	Gradient free		
DOF	$\underline{u}$	$\overline{u}$	$\underline{u}$	$\overline{u}$	$\underline{u}$	$\overline{u}$
5	0,1,1,0,1,0,1,1,1,0,1	1,0,0,1,0,1,0,0,0,1,0	0,1,1,0,1,0,1,1,1,0,1	1,0,0,1,0,1,0,0,0,1,0	0	0
6	0,0,0,0,1,0,1,1,1,0,1	1,1,1,0,0,1,0,0,0,1,0	0,0,0,0,1,0,1,1,1,0,1	1,1,1,1,0,1,0,0,0,1,0	0	1
7	0,0,1,0,1,0,1,1,1,0,1	1,0,0,1,0,1,0,0,0,1,0	0,1,1,0,1,0,1,1,1,0,1	1,0,0,1,0,1,0,0,0,1,0	1	0
8	0,0,1,1,0,1,0,0,0,0,1	1,1,0,0,1,0,1,1,1,1,0	0,0,1,1,0,1,0,0,0,0,1	1,1,0,0,1,0,1,1,1,1,0	0	0
11	0,0,1,1,0,1,0,0,0,0,1	1,1,0,0,1,0,1,1,1,1,0	0,0,1,1,0,1,0,0,0,0,1	1,1,0,0,1,0,1,1,1,1,0	0	0
12	0,0,0,0,1,0,1,1,1,1,0	1,1,1,1,0,0,0,0,0,0,1	0,0,0,0,1,0,1,1,1,1,0	1,1,1,1,0,1,0,0,0,0,1	0	1
15	0,0,1,1,0,1,0,0,0,0,1	1,0,0,0,1,0,1,1,1,1,0	0,0,1,1,0,1,0,0,0,0,1	1,1,0,0,1,0,1,1,1,1,0	0	1
16	1,0,0,1,0,1,0,0,0,1,0	0,1,1,0,1,0,1,1,1,0,1	1,0,0,1,0,1,0,0,0,1,0	0,1,1,0,1,0,1,1,1,0,1	0	0
17	0,1,1,1,1,1,0,1,0,0,1	1,0,0,0,0,0,1,0,1,1,0	0,1,1,1,1,1,0,1,0,0,1	1,0,0,0,0,0,1,0,1,1,0	0	0
18	0,0,0,0,0,0,1,1,1,0,1	1,1,1,1,1,1,0,0,0,1,0	0,0,0,0,0,0,1,1,1,0,1	1,1,1,1,1,1,0,0,0,1,0	0	0
19	0,1,1,1,1,1,0,0,0,0,1	1,0,0,0,0,0,1,1,1,1,0	0,1,1,1,1,1,0,0,0,0,1	1,0,0,0,0,0,1,1,1,1,0	0	0
20	0,0,1,0,0,1,0,0,0,0,1	1,1,0,1,1,0,1,1,1,1,0	0,0,1,0,0,1,0,0,0,0,1	1,1,0,1,1,0,1,1,1,1,0	0	0

2D example

	Combinatoric		Gradient free			
DOF	$\underline{u}$ $[m]$	$\overline{u}$ $[m]$	$\underline{u}$ $[m]$	$\overline{u}$ $[m]$	Error $\underline{u}$ %	Error $\overline{u}$ %
5	2.991120E-09	4.758950E-08	2.991120E-09	4.758950E-08	0.000000E+00	0.000000E+00
6	-3.401560E-07	-2.737550E-07	-3.401560E-07	-2.738240E-07	0.000000E+00	2.520502E-02
7	-2.700300E-08	3.178720E-08	-2.681040E-08	3.178720E-08	7.132541E-01	0.000000E+00
8	-4.259370E-07	-2.781400E-07	-4.259370E-07	-2.781400E-07	0.000000E+00	0.000000E+00
11	-4.758950E-08	-2.991120E-09	-4.758950E-08	-2.991120E-09	0.000000E+00	0.000000E+00
12	-3.401560E-07	-2.737550E-07	-3.401560E-07	-2.738240E-07	0.000000E+00	2.520502E-02
15	-3.178720E-08	2.700300E-08	-3.178720E-08	2.681040E-08	0.000000E+00	7.132541E-01
16	-4.259370E-07	-2.781400E-07	-4.259370E-07	-2.781400E-07	0.000000E+00	0.000000E+00
17	-8.584880E-09	1.665680E-07	-8.584880E-09	1.665680E-07	0.000000E+00	0.000000E+00
18	-5.498820E-07	-4.324890E-07	-5.498820E-07	-4.324890E-07	0.000000E+00	0.000000E+00
19	-2.279290E-08	1.656670E-07	-2.279290E-08	1.656670E-07	0.000000E+00	0.000000E+00
20	-1.091250E-06	-7.769350E-07	-1.091250E-06	-7.769350E-07	0.000000E+00	0.000000E+00

Veryfication of the results in ANSYS



Ordinary differential equations

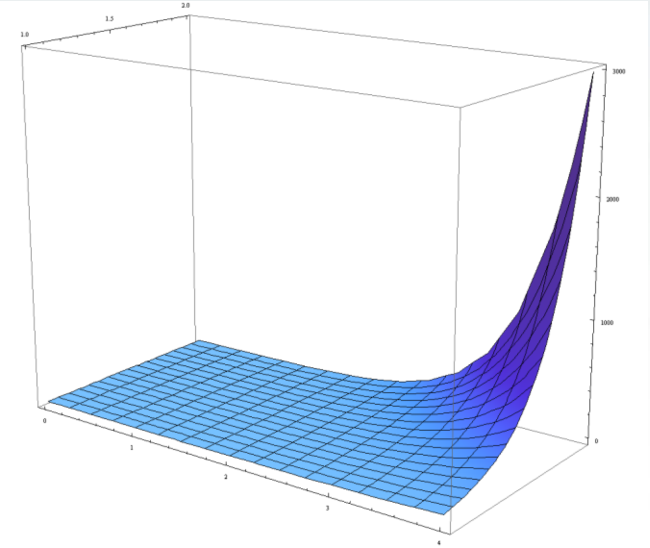
$$y' = py \qquad y(0) = 1$$

Solution

$$y = e^{pt}$$

Interval solution

$$\underline{y}(t) = e^{\underline{p}t} \qquad \overline{y}(t) = e^{\overline{p}t}$$



Sensitivity

$$y = e^{pt}$$

$$\frac{\partial y}{\partial p} = te^{pt}$$

Sensitivity without analytical solution

$$y' = py$$

$$\frac{\partial}{\partial p} y' = \frac{\partial}{\partial p}(py)$$

$$\frac{d}{dt}\left(\frac{\partial y}{\partial p}\right) = y + p\frac{\partial y}{\partial p}$$

$$\frac{d\mathrm{v}}{dt} = y + p\mathrm{v}$$

Sensitivity without analytical solution

$$\frac{d\mathrm{v}}{dt} = y + p\mathrm{v}$$

$$\mathrm{v}(0) = 0$$

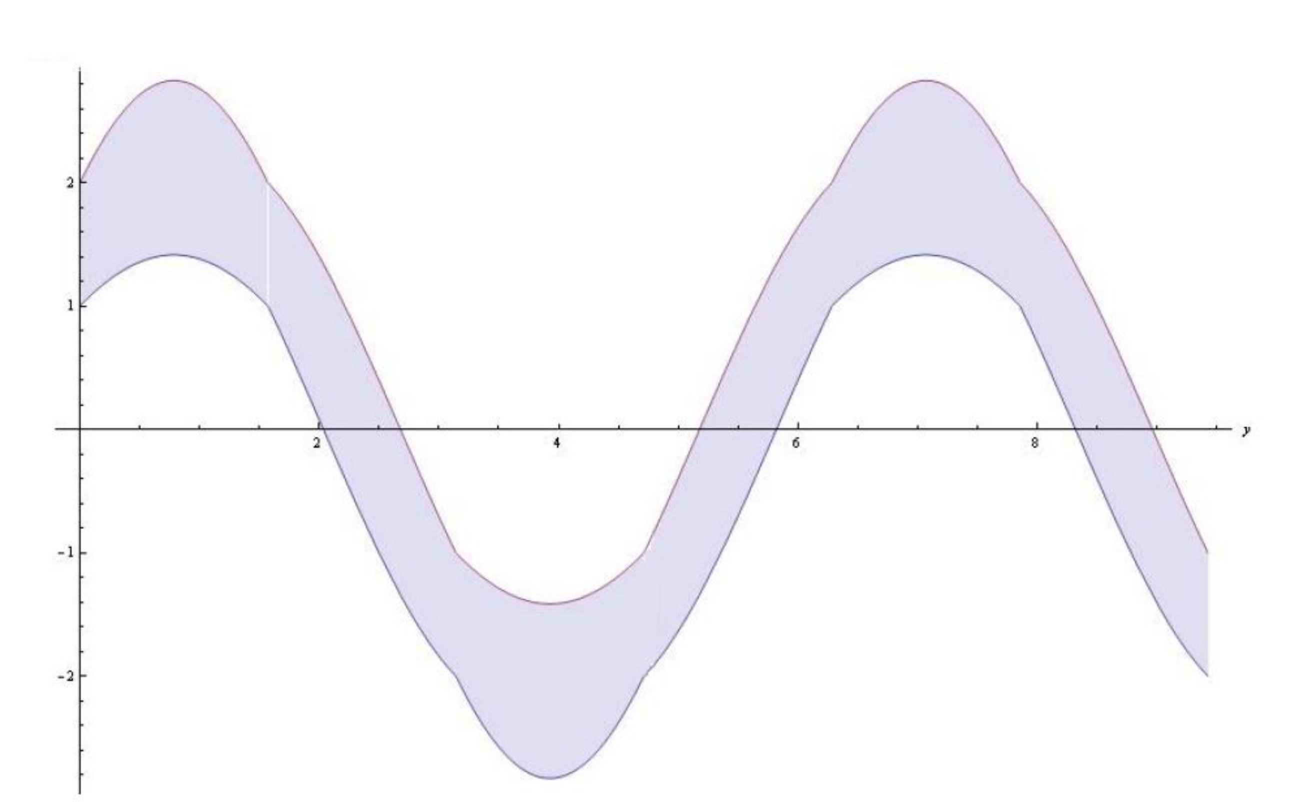
$$\frac{\partial y}{\partial p} = \mathrm{v} = te^{pt}$$

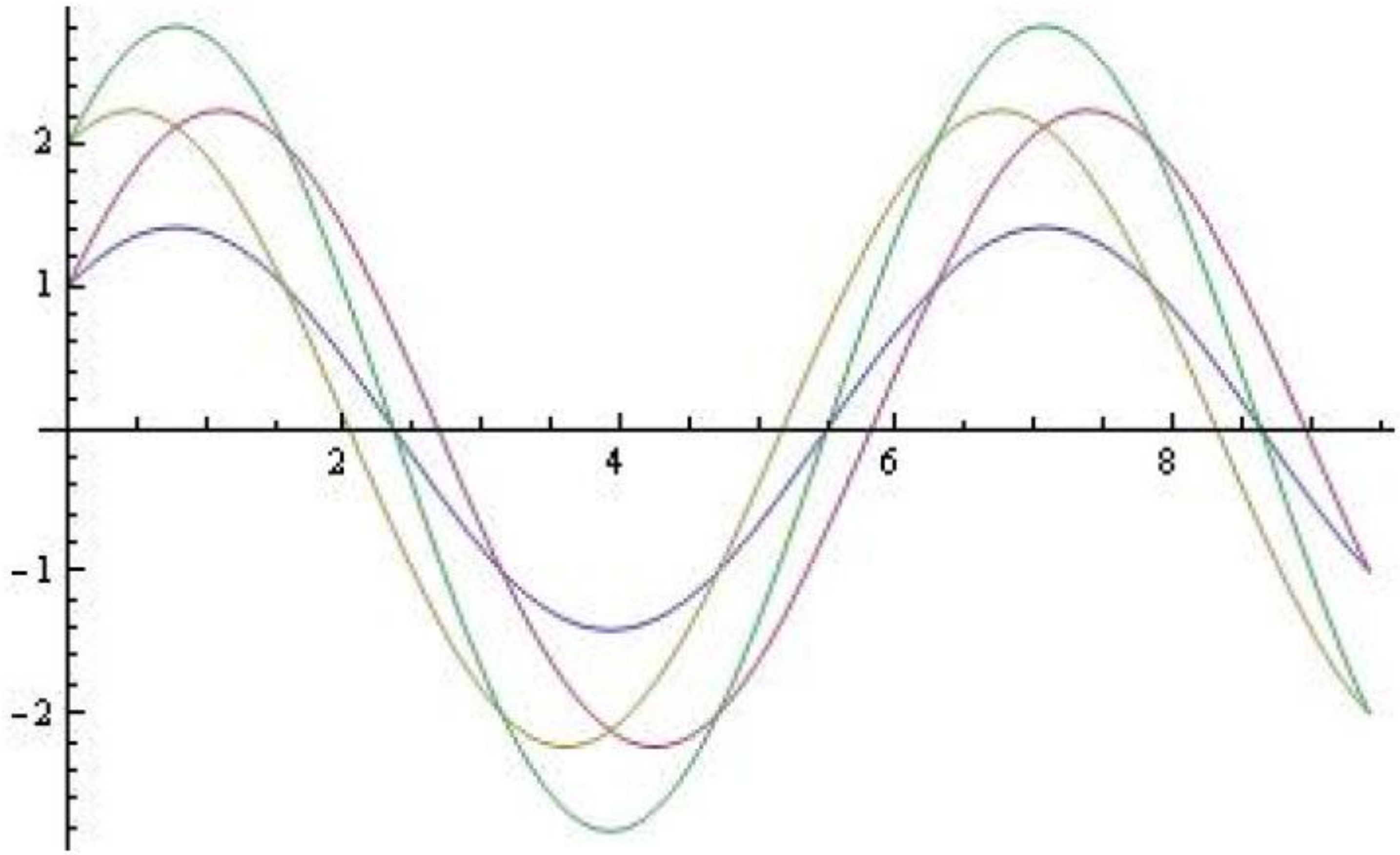
Second order equations

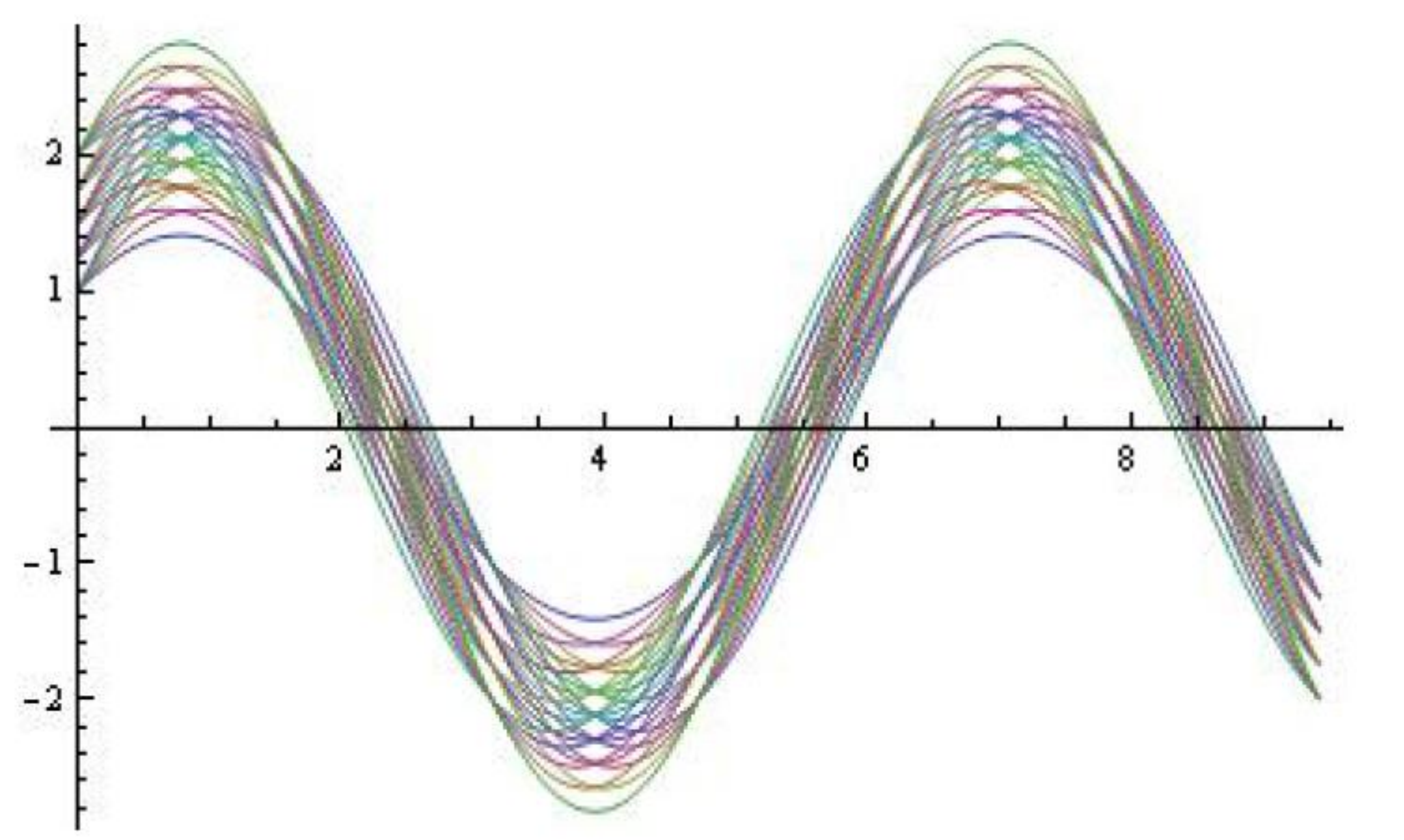
$\ddot{y} + \omega^2 y = 0$

$x_0 \in [x_0]$

$v_0 \in [v_0]$




2
1
-1
-2
2
4
6
8


2
1
-1
-2
2
4
6
8

Example

Let assume that $A \in \left[\underline{A}, \overline{A}\right] = [1,2]$, $B \in \left[\underline{B}, \overline{B}\right] = [1,2]$, $\omega = 1$, then

$$y = A\cos t + B\sin t$$

$$\frac{\partial y}{\partial A} = \cos t$$

$$\frac{\partial y}{\partial B} = \sin t$$

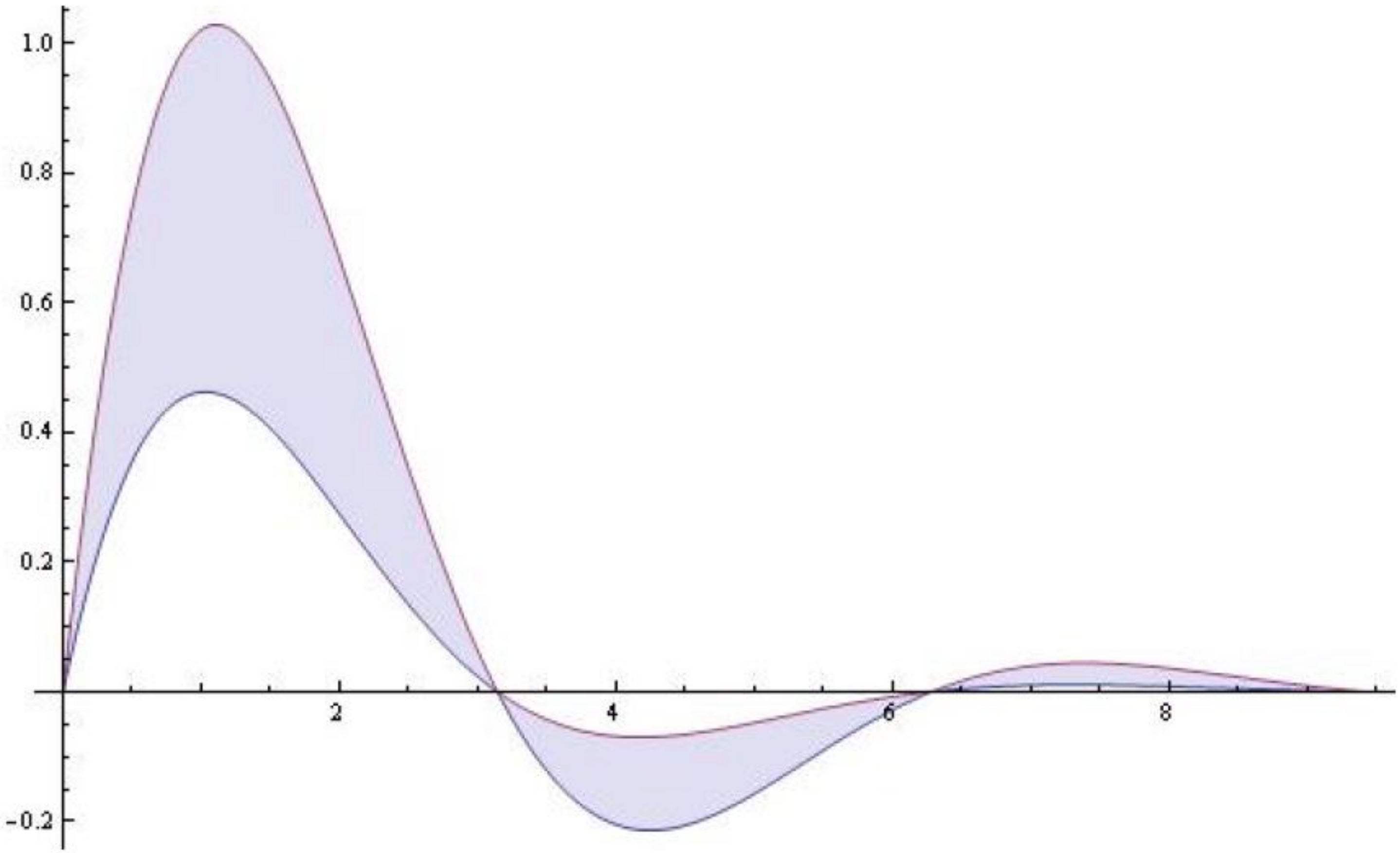
Example

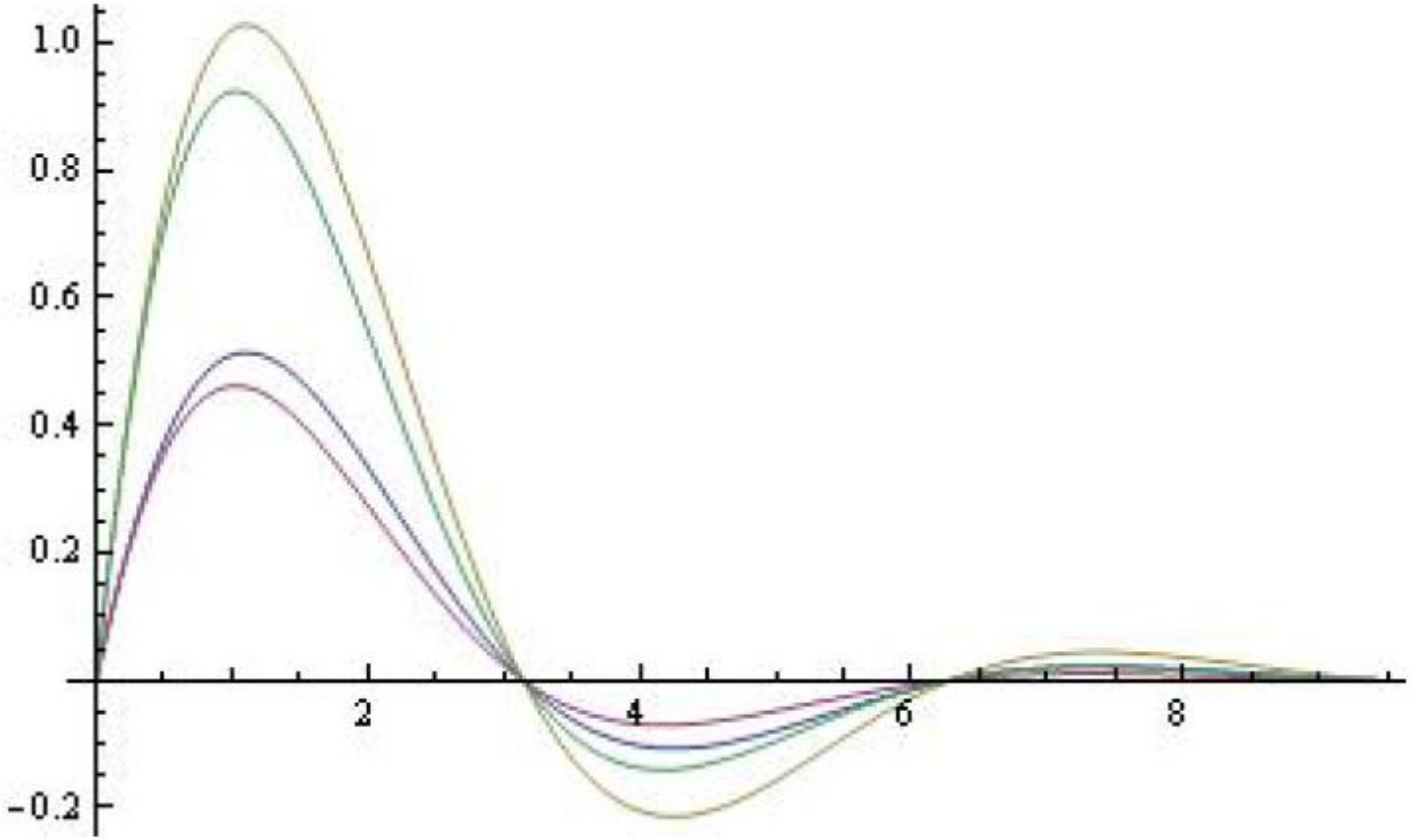
$$\underline{y}(t) = \begin{cases} 1\cos(t) + 1\sin(t), t \in \left[0, \frac{\pi}{2}\right] \\ 2\cos(t) + 1\sin(t), t \in \left[\frac{\pi}{2}, \pi\right] \\ 2\cos(t) + 2\sin(t), t \in \left[\pi, \frac{3\pi}{2}\right] \\ 1\cos(t) + 2\sin(t), t \in \left[\frac{3\pi}{2}, 2\pi\right] \end{cases},$$

Example

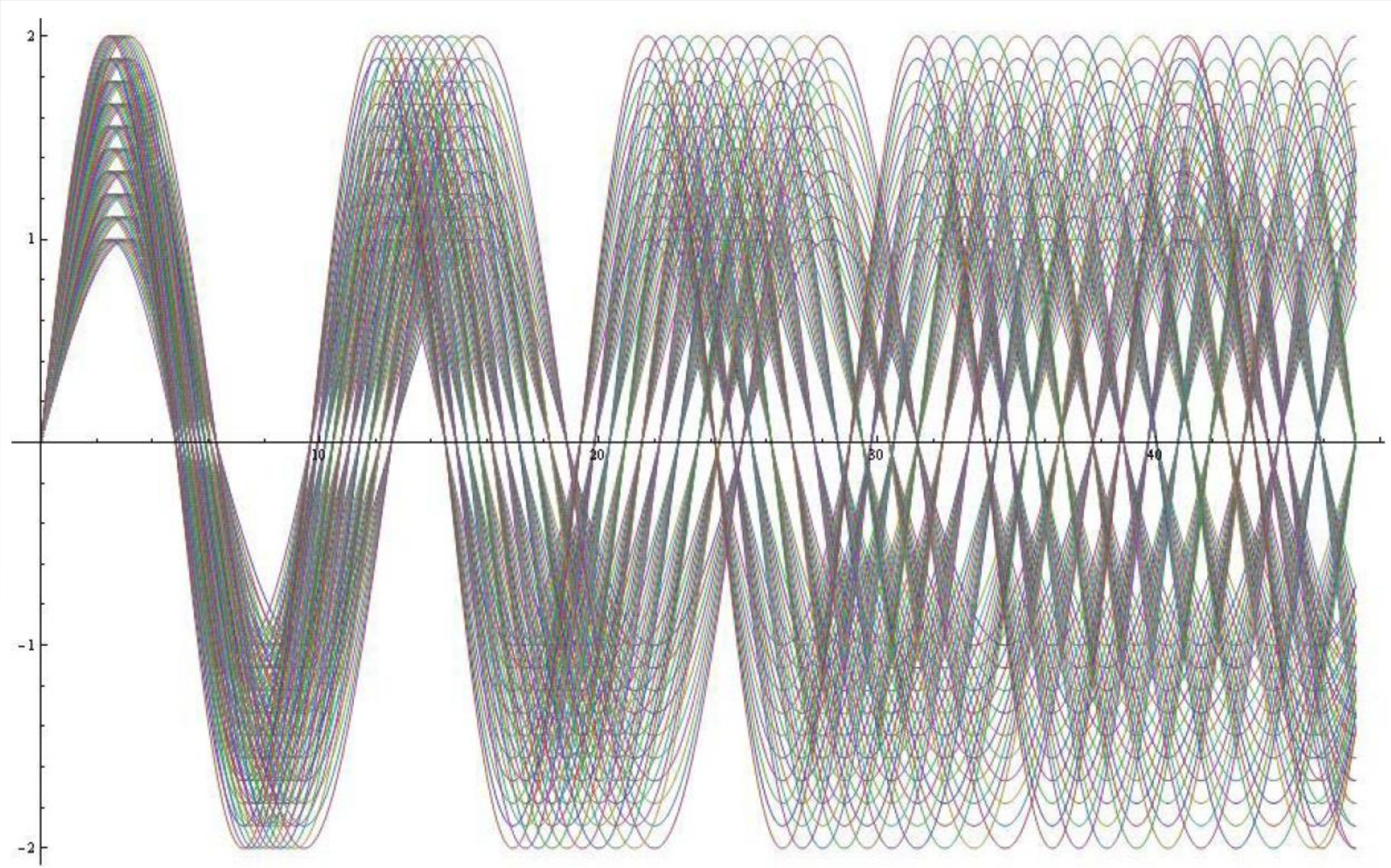
$$\overline{y}(t)=\begin{cases}2\cos(t)+2\sin(t), t\in\left[0,\frac{\pi}{2}\right]\\ 1\cos(t)+2\sin(t), t\in\left[\frac{\pi}{2},\pi\right]\\ 1\cos(t)+1\sin(t), t\in\left[\pi,\frac{3\pi}{2}\right]\\ 2\cos(t)+1\sin(t), t\in\left[\frac{3\pi}{2},2\pi\right]\end{cases}.$$

$$\ddot{y} + 2\beta\dot{y} + \omega_0^2 y = 0$$




1.0
0.8
0.6
0.4
0.2
−0.2
2
4
6
8

$$y = A \sin \omega t$$



General form of the interval solution

$$y(t) \in \left[y\left(t, p^{\min}(t)\right), y\left(t, p^{\max}(t)\right) \right]$$

Thank you