Numerical solutions of fuzzy partial differential equations and its applications in computational mechanics

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Abstract

Calculation of the solution of fuzzy partial differential equations is in general very difficult. We can find the exact solution only in some special cases. Fortunately, in most of engineering applications relations between the solutions and uncertain parameters are monotone (we can assume that, when the uncertainty of the parameters is sufficiently small). In this case, the exact solution can be calculated using only endpoints of given intervals. In order to improve the efficiency of calculation we can apply sensitivity analysis.

In this paper, a very efficient algorithm of solution was presented. This algorithm is based on finite element method (or any other numerical method of solution PDE like for example FEM or BEM) and sensitivity analysis. Using this method we can solve engineering problems with thousands degree of freedom. Fuzzy partial differential equations can be applied for modeling of mechanical system (structures) with uncertain parameters.

To construct the fuzzy membership function random sets can be applied. This theory contains fuzzy sets and probability theory as special cases. Using algorithms, which are described in this paper we can solve partial differential equations with random and fuzzy parameters.

Keywords: fuzzy sets, random sets, interval arithmetic, fuzzy partial differential equations.

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1 Introduction

Fuzzy number is a fuzzy set F of the real line R, which is convex and normal. Let $F(R)$ denote the set of all fuzzy numbers, which are upper semicontinuous and have compact support. If $F \in F(R)$

$$\mu_F : R \ni x \to \mu_F(x) \in [0,1] \subset R \qquad (1)$$

we can also write

$$\mu(.\,|\,.) : R \times F(R) \ni (x, F) \to \mu(x\,|\,F) \in [0,1] \subset R \qquad (2)$$

2 Fuzzy equation

Let us consider the following equations with fuzzy parameter

$$\frac{du}{dx} = hx, u(0) = u_0 . \qquad (3)$$

An analytical solution of this equation is the following

$$u(x,h) = \frac{hx^2}{2} + u_0 . \qquad (4)$$

The membership function of the fuzzy solution $u_F(x) \in F(R)$ can be calculated using the extension principle

$$\mu(\xi\,|\,u_F(x)) = \sup_{h:\xi = \frac{hx^2}{2} + u_0} \mu_F(h) . \qquad (5)$$

Let us consider some partial differential equations with vector of fuzzy parameters $\mathbf{h} \in F$

$$\mathbf{H}\left(\mathbf{x}, \mathbf{h}, \mathbf{u}, \frac{\partial \mathbf{u}}{\partial \mathbf{x}}, \ldots, \frac{\partial^k \mathbf{u}}{\partial \mathbf{x}^k}\right) = 0, \quad \mathbf{u} \in V \tag{6}$$

where V is some functional space. If we know the exact solution of the problem (6) $\mathbf{u} = \mathbf{u}(\mathbf{x}, \mathbf{h})$ we can calculate the fuzzy solution using the extension principle:

$$\mu(\xi \mid \mathbf{u}_F(\mathbf{x})) = \sup_{\mathbf{h}:\, \xi = \mathbf{u}(\mathbf{x}, \mathbf{h})} \mu_F(\mathbf{h}) \tag{7}$$

The same solution can be calculated using α-level cut method (Buckley, Qy 1990, Buckley, Feuring 2000). The algorithm is the following:

Algorithm 1

1) Calculate α-level cut of fuzzy parameters $\mathbf{h} \in F$

$$\hat{\mathbf{h}}_\alpha = \{\mathbf{h} : \mu_F(\mathbf{h}) \geq \alpha\}. \tag{8}$$

2) Calculate the solution of partial differential equations with interval parameters:

$$\hat{\mathbf{u}}_\alpha(\mathbf{x}) = \{\mathbf{u}(\mathbf{x}, \mathbf{h}) : \mathbf{h} \in \hat{\mathbf{h}}_\alpha\}. \tag{9}$$

3) Calculate fuzzy membership function of the solution:

$$\mu(\xi \mid \mathbf{u}_F(\mathbf{x})) = sup\{\alpha : \xi \in \hat{\mathbf{u}}_\alpha\}. \tag{10}$$

The most difficult part of this algorithm is the step 2.

3 Random sets interpretation of fuzzy

Let us consider probability space $(\Omega, P_\Omega, \Sigma_\Omega)$ and interval-valued random variable:

$$\hat{H}_\Omega : \Omega \ni \omega \rightarrow \hat{H}_\Omega \in I(R) \tag{11}$$

where $I(R)$ is a set of all intervals.
Using such random variable, we can define upper and lower probability

$$Pl(A) = P_\Omega\{\omega : \hat{H}_\Omega(\omega) \cap A \neq \emptyset\} \tag{12}$$

$$Bel(A) = P_\Omega\{\omega : \hat{H}_\Omega(\omega) \subseteq A\} \tag{13}$$

Let us consider discrete random variable, which satisfy the following condition:

$$\hat{H}_\Omega(\omega_1) \supseteq \hat{H}_\Omega(\omega_2) \supseteq ... \supseteq \hat{H}_\Omega(\omega_n) \tag{14}$$

then we can define fuzzy membership function in the following way:

$$\mu_F(h) = P_\Omega\{\omega : h \in \hat{H}_\Omega(\omega)\} \tag{15}$$

Let us consider some mechanical system and performance function $g(\mathbf{h})$, which has the following properties:
- if $g(\mathbf{h}) \geq 0$, then the structure is safe,
- if $g(\mathbf{h}) < 0$, then the structure failed.

Upper probability of failure of the structure can be defined in the following way:

$$P_f^+ = P_\Omega\{\omega : g(\hat{H}_\Omega(\omega)) \cap (-\infty, 0) \neq \emptyset\} \tag{16}$$

If the structure has fuzzy parameters $\mathbf{h} \in F$, then we can calculate upper probability of failure in the following way:

$$P_f^+ = \sup_{\mathbf{h}:g(\mathbf{h})<0} \mu_F(\mathbf{h}) \tag{17}$$

Upper probability of failure of the structures with random ($\mathbf{X}_\Omega : \Omega \ni \omega \to \mathbf{X}_\Omega(\omega) \in R^n$) and fuzzy ($\mu_F : R^m \in \mathbf{h} \to \mu_F(\mathbf{h}) \in R$) parameters (i.e. $y = g(\mathbf{x},\mathbf{h})$) can be calculated using the following formula

$$P_f^+ = \sum_{\mathbf{x}} \mu_{\mathbf{x}(F)}(\mathbf{x}) P_\Omega\{\mathbf{x}\} = E_\Omega\{\mu_{\mathbf{x}(F)}(\mathbf{x})\} \tag{18}$$

where

$$\mu_{\mathbf{x}(F)}(\mathbf{x}) = \sup_{\mathbf{h}:g(\mathbf{x},\mathbf{h})<0} \mu_F(\mathbf{h}) \tag{19}$$

4 Numerical methods of solution of partial differential equations

Many problems in engineering can be described using partial differential equations particularly:

- static and dynamic of structures,
- biomechanics,
- heat and mass transfer,
- electromagnetic fields,
- meteorology etc.

The most popular methods of solution of such equations are:

- finite element method (FEM),
- boundary element method (BEM),
- finite difference method (FDM).

Most universal and popular is finite element method (Ciarlet 1978). The FEM algorithm has the following steps:
1) Formulate boundary value problem

$$\mathbf{L}(\mathbf{x},\mathbf{u}) = \mathbf{f}(\mathbf{x}), \quad \mathbf{u} \in V \tag{20}$$

where $\mathbf{L}(\mathbf{x},.)$ is differential operator, $\mathbf{f}(\mathbf{x})$ is some function and V is a functional space (e.g. Sobolev space).
2) Formulate variational equation of the problem

$$\forall \mathbf{v} \in V, \quad a(\mathbf{u},\mathbf{v}) = l(\mathbf{v}) \tag{21}$$

3) Discretize domain Ω using finite elements $\bigcup_e \Omega^e = \Omega$ and build a solution space V_h

$$\forall \mathbf{u}_h \in V_h, \exists \mathbf{u} \in R^N, \mathbf{u}_h(\mathbf{x}) = \mathbf{N}(\mathbf{x})\mathbf{u} \tag{22}$$

where $\mathbf{N}(\mathbf{x}) = \begin{bmatrix} N_1^1(x) & \dots & N_1^N(x) \\ \dots & \dots & \dots \\ N_n^1(x) & \dots & N_n^N(x) \end{bmatrix}$ is a matrix of shape functions.

An approximate solution has the following form:

$$u_i^e(\mathbf{x}) = \sum_j N_{ij}^e(\mathbf{x}) \cdot u_j^e, \quad \mathbf{x} \in \Omega^e \quad (\text{i.e.}\ \mathbf{u}_h^e(\mathbf{x}) = \mathbf{N}^e(\mathbf{x})\mathbf{u}^e) \tag{23}$$

$$u_i^h(\mathbf{x}) = \sum_j N_{ij}(\mathbf{x}) \cdot u_j, \quad \mathbf{x} \in \Omega \qquad (\text{i.e.}\ \mathbf{u}_h(\mathbf{x}) = \mathbf{N}(\mathbf{x})\mathbf{u}) \tag{24}$$

4) The approximate solution satisfies the following variational equation:

$$\forall \mathbf{v}_h \in V_h, \quad a(\mathbf{u}_h, \mathbf{v}_h) = l(\mathbf{v}_h) \tag{25}$$

5) Vector of nodal solution can be calculated as a solution of the following system of linear equation

$$\mathbf{Ku} = \mathbf{Q} \tag{26}$$

where **K** (stiffness matrix) and the vector **Q** are defined in the following way:

$$K_{ij} = a(\mathbf{N}_i, \mathbf{N}_j) \tag{27}$$

$$Q_i = l(\mathbf{N}_i) \tag{28}$$

6) If we know, the solution of the system of equation (26) **u** (vector of nodal solution) we can calculate the value of approximate solution between the nodes using equations (23, 24).

Finite element method was implemented in many commercial engineering programs e.g.:

- ABACUS (http://www.hks.com/)
- ADINA (http://www.adina.com/)
- ANSYS (http://www.ansys.com/)
- etc.

Using this method we can solve problems with thousands or even millions degree of freedom.

5 Numerical methods of solution of the fuzzy partial differential equations

To solution of the fuzzy partial differential equations, we can also apply algorithm of the finite element method. In this case, equation (26) has the following form:

$$\mathbf{K}(\mathbf{h})\mathbf{u} = \mathbf{Q}(\mathbf{h}), \quad \mathbf{h} \in \hat{\mathbf{h}}_\alpha \tag{29}$$

or in nonlinear problems

$$\mathbf{K}(\mathbf{h},\mathbf{u})\mathbf{u} = \mathbf{Q}(\mathbf{h}), \quad \mathbf{h} \in \hat{\mathbf{h}}_\alpha \tag{30}$$

If we know the solution $\hat{\mathbf{u}}_\alpha$ of parameter dependent system of equation (29)

$$\hat{\mathbf{u}}_\alpha = hull\{\mathbf{u} : \mathbf{K}(\mathbf{h})\mathbf{u} = \mathbf{Q}(\mathbf{h}), \mathbf{h} \in \hat{\mathbf{h}}_\alpha\} \tag{31}$$

(the symbol $hull\, S$ denote the smallest interval, which contain the set S) we can calculate a fuzzy membership function of the fuzzy nodal solution $\mathbf{u}_F$ in the following way:

$$\mu(\mathbf{u} \mid \mathbf{u}_F) = sup\{\alpha : \mathbf{u} \in \hat{\mathbf{u}}_\alpha\} \tag{32}$$

The exact solution set $\{\mathbf{u} : \mathbf{K}(\mathbf{h})\mathbf{u} = \mathbf{Q}(\mathbf{h}), \mathbf{h} \in \hat{\mathbf{h}}_\alpha\}$ of the problem (29) is very complicated because of this in applications we use only the smallest interval which contain the exact solution i.e. $hull\{\mathbf{u} : \mathbf{K}(\mathbf{h})\mathbf{u} = \mathbf{Q}(\mathbf{h}), \mathbf{h} \in \hat{\mathbf{h}}_\alpha\}$ (Kulpa et all 1998).

The fuzzy solution between the nodes $\mathbf{u}_F(\mathbf{x})$ can be calculated using the following formula:

$$\mu(\mathbf{u} \mid \mathbf{u}_F(\mathbf{x})) = sup\{\alpha : \mathbf{u} \in \hat{\mathbf{u}}_\alpha(\mathbf{x})\} \tag{33}$$

where $\hat{\mathbf{u}}_\alpha(\mathbf{x})$ is defined as follows

$$\hat{\mathbf{u}}_\alpha(\mathbf{x}) = hull\{\mathbf{u}_h(\mathbf{x},\mathbf{h}) : \mathbf{h} \in \hat{\mathbf{h}}_\alpha\} \tag{34}$$

i.e.

$$\hat{\mathbf{u}}_\alpha(\mathbf{x}) = hull\{\mathbf{N}(\mathbf{x})\mathbf{u} : \mathbf{K}(\mathbf{h})\mathbf{u} = \mathbf{Q}(\mathbf{h}),\ \mathbf{h} \in \hat{\mathbf{h}}_\alpha\} \tag{35}$$

The finite difference method and the boundary element method can be also applied to calculation of numerical solution of the fuzzy partial differential equation. With all mentioned methods, we can apply the following general algorithm.

Algorithm 2

1) Calculate α-cut of fuzzy parameters $\hat{\mathbf{h}}_\alpha = \{\mathbf{h} : \mu_F(\mathbf{h}) \geq \alpha\}$.

2) Solve system of parameter dependent system of equation

$$\hat{\mathbf{u}}_\alpha = hull\{\mathbf{u} : \mathbf{K}(\mathbf{h})\mathbf{u} = \mathbf{Q}(\mathbf{h}), \mathbf{h} \in \hat{\mathbf{h}}_\alpha\} \tag{36}$$

3) Calculate fuzzy nodal solution $\mathbf{u}_F$

$$\mu(\mathbf{u} \mid \mathbf{u}_F) = sup\{\alpha : \mathbf{u} \in \hat{\mathbf{u}}_\alpha\} \tag{37}$$

4) Calculate fuzzy solution between the nodal points $\mathbf{u}_F(\mathbf{x})$

$$\mu(\mathbf{u} \mid \mathbf{u}_F(\mathbf{x})) = sup\{\alpha : \mathbf{u} \in \hat{\mathbf{u}}_\alpha(\mathbf{x})\} \tag{38}$$

6 Systems of algebraic equations with interval parameters

The most difficult part of second algorithm is the step 2. It can be shown that finding the solution of system of a linear interval equation is NP-hard (Kreinovich et all 1998). Because of that the interval solution $\hat{\mathbf{u}}_\alpha$, which is defined in the equation (36) can be found only in special cases.

6.1 Application of monotone functions

Many numerical examples shows, that the relation between the solution **u**=**u**(**h**) and uncertain parameters **h** is monotone (McWilliam 2000, Noor et all 2000, Pownuk 2000).

Let us consider function $y=u(h)$ and the interval $\hat{h}=[h^-,h^+]$. If function $u(h)$ is monotone, then the extreme values of the function u over the interval $\hat{h}$ can be calculated using only the endpoints h^-,h^+.

$$\hat{y}=[y^-,y^+]=[min\{u(h^-),u(h^+)\},max\{u(h^-),u(h^+)\}] \quad (39)$$

where

$$y^- = \inf_{h:h\in\hat{h}} u(h),\quad y^+ = \sup_{h:h\in\hat{h}} u(h). \quad (40)$$

Let's assume that the function $\mathbf{u}$ depends on m parameters $h_1,...,h_m$ (i.e. $\mathbf{u}:R^m \ni \mathbf{h}\rightarrow\mathbf{u}(\mathbf{h})\in R^n$), which belong to the m intervals $\hat{h}_1,...,\hat{h}_m$ (i.e. $\mathbf{h}\in\hat{\mathbf{h}}$). If this function is monotone then the extreme values can be found after calculation all combination of endpoints of the multidimensional interval $\hat{\mathbf{h}}$.

$$u_i^- = min\{u_i(\mathbf{h}) : \mathbf{h} \in Vertex(\hat{\mathbf{h}})\}, \quad (41)$$

$$u_i^+ = max\{u_i(\mathbf{h}) : \mathbf{h} \in Vertex(\hat{\mathbf{h}})\}. \quad (42)$$

where $Vertex(\hat{\mathbf{h}})$ is a set of all vertex of the interval $\hat{\mathbf{h}}$. Now we can write

$$\hat{\mathbf{u}} = [u_1^-, u_1^+] \times [u_2^-, u_2^+] \times ... \times [u_n^-, u_n^+] = hull\, \mathbf{u}(\hat{\mathbf{h}}) \quad (43)$$

To calculation of the vector $\hat{\mathbf{u}}$ we have to calculate the value of the function $\mathbf{u}(\mathbf{h})$ 2^m times. In the same way, we can calculate the solution of the problem (36)

$$u_i^- = min\{u_i : \mathbf{K}(\mathbf{h})\mathbf{u} = \mathbf{Q}(\mathbf{h}), \mathbf{h} \in Vertex(\hat{\mathbf{h}})\}, \quad (44)$$

$$u_i^+ = max\{u_i : \mathbf{K}(\mathbf{h})\mathbf{u} = \mathbf{Q}(\mathbf{h}), \mathbf{h} \in Vertex(\hat{\mathbf{h}})\}. \quad (45)$$

Unfortunately, this method has very high computational complexity and cannot be applied to problems that are more complicated.

6.2 Application of sensitivity analysis

Let us consider a function $u : R \ni h \rightarrow u(h) \in R$. If the derivative $\frac{\partial u}{\partial h}$ has constant sign then extreme values can be calculated using the following formulas

$$\text{If } \frac{\partial u(h_0)}{\partial h} > 0, \text{ then } u^- = u(h^-),\ u^+ = u(h^+) \quad (46)$$

$$\text{If } \frac{\partial u(h_0)}{\partial h} < 0, \text{ then } u^- = u(h^+),\ u^+ = u(h^-) \quad (47)$$

where $h_0 = \dfrac{h^- + h^+}{2} = mid(\hat{h})$.

In multidimensional case in order to calculate extreme values of function **u**=**u**(**h**) ($u_i = u_i(\mathbf{h})$) we can compute the sign vector $\mathbf{S}^i$

$$\mathbf{S}^i = \left[sign\left(\frac{\partial u_i(\mathbf{h}_0)}{\partial h_1}\right) \quad \ldots \quad sign\left(\frac{\partial u_i(\mathbf{h}_0)}{\partial h_m}\right)\right]^T \tag{48}$$

$$\mathbf{S}^i = [S_1^i, \ldots, S_m^i]^T \tag{49}$$

where $\mathbf{h}_0 = \left[\dfrac{h_1^- + h_1^+}{2} \quad \ldots \quad \dfrac{h_m^- + h_m^+}{2} \right]^T = mid(\hat{\mathbf{h}})$.

If function $u_i = u_i(\mathbf{h})$ is monotone, then upper bound can be calculated using the following point

$$\mathbf{h}_i^{upper} = \left[{}^i h_1^{upper} \quad \ldots \quad {}^i h_m^{upper} \right]^T \tag{50}$$

where

$$\text{if } S_k^i > 0 \text{ then } {}^i h_k^{upper} = h_k^+ \tag{51}$$

$$\text{if } S_k^i < 0 \text{ then } {}^i h_k^{upper} = h_k^- . \tag{52}$$

In this case $u_i^+ = u_i(\mathbf{h}_i^{upper})$. In the same way, we can construct lower bound of the function $u_i = u_i(\mathbf{h})$.

$$\mathbf{h}_i^{lower} = \left[{}^i h_1^{lower} \quad \ldots \quad {}^i h_m^{lower} \right]^T \tag{53}$$

where

$$\text{if } S_k^i > 0 \text{ then } {}^i h_k^{upper} = h_k^- \tag{54}$$

$$\text{if } S_k^i < 0 \text{ then } {}^i h_k^{upper} = h_k^+ . \tag{55}$$

and finally

$$u_i^- = u_i(\mathbf{h}_i^{lower}), \quad u_i^+ = u_i(\mathbf{h}_i^{upper}), \tag{56}$$

$$\hat{\mathbf{u}} = [u_1^-, u_1^+] \times ... \times [u_m^-, u_m^+]. \tag{57}$$

We see that extreme values of the function $\mathbf{u} = \mathbf{u}(\mathbf{h})$ can be calculated using the sign vectors $\mathbf{S}^i$ and the endpoints of interval $\hat{\mathbf{h}}$

$$\mathbf{h}_i^{lower} = \mathbf{h}^{lower}(\mathbf{S}^i, \hat{\mathbf{h}}), \quad \mathbf{h}_i^{upper} = \mathbf{h}^{upper}(\mathbf{S}_i, \hat{\mathbf{h}}) \tag{58}$$

$$\hat{u}_i = [u_i^-, u_i^-] = [u\left(\mathbf{h}^{lower}(\mathbf{S}_i, \hat{\mathbf{h}})\right), u\left(\mathbf{h}^{upper}(\mathbf{S}^i, \hat{\mathbf{h}})\right)] \tag{59}$$

The vector $\mathbf{S}^i$ have to be calculated for each coordinate of the vector **u** i.e. *n* times. From the definition of the vectors $\mathbf{S}^i$ arise that

$$\mathbf{h}^{lower}(\mathbf{S}^i, \hat{\mathbf{h}}) = \mathbf{h}^{lower}((-1) \cdot \mathbf{S}^i, \hat{\mathbf{h}}) \tag{60}$$

$$\mathbf{h}^{upper}(\mathbf{S}^i, \hat{\mathbf{h}}) = \mathbf{h}^{upper}((-1) \cdot \mathbf{S}^i, \hat{\mathbf{h}}) \tag{61}$$

i.e. the sign vector $\mathbf{S}^i$ generate the same lower and upper bound as vector $(-1) \cdot \mathbf{S}^i$. From computational point of view, it is convenient to find independent sign vectors, which generate different lower and upper bounds of the solution.

$$IndS = \{\mathbf{S}^{1*},...,\mathbf{S}^{k*}\} \quad (62)$$

$$\forall \mathbf{S}^{i*}, \mathbf{S}^{j*} \in IndS, (i \neq j) \Rightarrow ((\mathbf{S}^{j*} \neq \mathbf{S}^{i*}) \wedge (\mathbf{S}^{i*} \neq (-1)\cdot \mathbf{S}^{j*})) \quad (63)$$

According to my experience, number of the sign vectors $\mathbf{S}^{j*}$ is much lower than number of the vectors $\mathbf{S}^{i}$ (Pownuk 2001).
Now we can apply sensitivity analysis method to solution of the problem (36). Whole algorithm has the following steps

Algorithm 3
1) Formulate parameter dependent system of equation with interval parameters in the form (36). And calculate $\mathbf{u}(\mathbf{h}_{\alpha}^{0})$

$$\mathbf{K}(\mathbf{h}_{\alpha}^{0})\mathbf{u}(\mathbf{h}_{\alpha}^{0}) = \mathbf{Q}(\mathbf{h}_{\alpha}^{0}), \quad \mathbf{h}_{\alpha}^{0} = mid(\hat{\mathbf{h}}_{\alpha}) \quad (64)$$

2) For i=1,...,m calculate $\dfrac{\partial \mathbf{u}(\mathbf{h}_{\alpha}^{0})}{\partial h_i}$, where

$$\mathbf{K}(\mathbf{h}_{\alpha}^{0})\frac{\partial \mathbf{u}(\mathbf{h}_{\alpha}^{0})}{\partial h_i} = \frac{\partial \mathbf{Q}(\mathbf{h}_{\alpha}^{0})}{\partial h_i} - \frac{\partial \mathbf{K}(\mathbf{h}_{\alpha}^{0})}{\partial h_i}\mathbf{u}(\mathbf{h}_{\alpha}^{0}). \quad (65)$$

3) For i=1,...,n (n – number of degree of freedom) calculate the sign vector

$$\mathbf{S}_{\alpha}^{i}\cdot = \left[sign\left(\frac{\partial u_i(\mathbf{h}_{\alpha}^{0})}{\partial h_1}\right) \quad ... \quad sign\left(\frac{\partial u_i(\mathbf{h}_{\alpha}^{0})}{\partial h_m}\right) \right] \quad (66)$$

4) Calculate independent sign vectors $IndSign_{\alpha} = \{\mathbf{S}_{\alpha}^{1*},...,\mathbf{S}_{\alpha}^{k*}\}$ using condition (63) and create vector **U** such, that

$$\mathbf{S}_{\alpha}^{i} = \mathbf{S}_{\alpha}^{j*}, \text{ where } j = U_i \tag{67}$$

5) For i=1,...,k calculate interval solution $\hat{\mathbf{u}}_{\alpha}^{i*}$

$$\hat{\mathbf{u}}_{\alpha}^{i*} = [\mathbf{u}\left(\mathbf{h}^{lower}(\mathbf{S}_{\alpha}^{i*}, \hat{\mathbf{h}})\right), \mathbf{u}\left(\mathbf{h}^{upper}(\mathbf{S}_{\alpha}^{i*}, \hat{\mathbf{h}})\right)] \tag{68}$$

$$IndSolutin_{\alpha} = \{\hat{\mathbf{u}}_{\alpha}^{1*}, ..., \hat{\mathbf{u}}_{\alpha}^{k*}\} \tag{69}$$

6) Calculate extreme interval solution $\hat{\mathbf{u}}_{\alpha}$.
For i = 1,...,n

$$\hat{u}_{\alpha i} = [u_{\alpha i}^{j*-}, u_{\alpha i}^{j*-}], \text{ where } j = U_i \tag{70}$$

Computational complexity of this algorithm:
- step 1 – 1 solution systems of equations,
- step 2 – m solution systems of equations,
- step 5 – $2 \cdot k$ solution systems of equations ($1 \leq k \leq n$).

In presented algorithm we have to calculate a system of equation between $1 + m + 2$ and $1 + m + 2 \cdot n$ times.

6.3 Calculation of the solution between the nodes

Sometimes we would like to know the interval solution $\hat{\mathbf{u}}_{\alpha}(\mathbf{x})$ of the boundary value problem in the point $\mathbf{x} \in \Omega$ between the nodal points. If we assume that the function $u_i = u_i(\mathbf{x}, \mathbf{h})$ is monotone (for fixed $\mathbf{x} \in \Omega$), then to calculation of extreme values sensitivity analysis can be applied.

First, we have to calculate sensitivity vector $\mathbf{S}_{\alpha}^{\mathbf{x}}$

$$\mathbf{S}_{\alpha}^{\mathbf{x}} = \left[sign\left(\frac{\partial u_i(\mathbf{x}, \mathbf{h}_{\alpha}^{0})}{\partial h_1}\right) \quad ... \quad sign\left(\frac{\partial u_i(\mathbf{x}, \mathbf{h}_{\alpha}^{0})}{\partial h_m}\right) \right]. \tag{71}$$

From the equation (24) arise that:

$$\frac{\partial u_i(\mathbf{x},\mathbf{h}_\alpha^0)}{\partial h_q} = \frac{\partial N_{ij}(\mathbf{x},\mathbf{h}_\alpha^0)}{\partial h_q} u_j(\mathbf{h}_\alpha^0) + N_{ij}(\mathbf{x},\mathbf{h}_\alpha^0) \frac{\partial u_j(\mathbf{h}_\alpha^0)}{\partial h_q} \quad (72)$$

The vectors $\mathbf{u}(\mathbf{h}_\alpha^0)$, $\frac{\partial \mathbf{u}(\mathbf{h}_\alpha^0)}{\partial h_j}$ were calculated in the algorithm 3, because of that to calculation of $\frac{\partial u_i(\mathbf{x},\mathbf{h}_\alpha^0)}{\partial h_q}$ we don't have to solve any system of linear equation.

Now we have to check if the sign vector $\mathbf{S}_\alpha^{\mathbf{x}}$ is unique.

We assume that the sign vector $\mathbf{S}_\alpha^{\mathbf{x}}$ is unique if

$$\forall \mathbf{S}_\alpha^{i*} \in IndSign_\alpha,\ (\mathbf{S}_\alpha^{i*} \neq \mathbf{S}_\alpha^{\mathbf{x}}) \wedge (\mathbf{S}_\alpha^{i*} \neq (-1)\cdot \mathbf{S}_\alpha^{\mathbf{x}}) \quad (73)$$

If the sign vector $\mathbf{S}_\alpha^{\mathbf{x}}$ is not unique i.e.

$$\exists \mathbf{S}_\alpha^{p*} \in IndSign_\alpha,\ (\mathbf{S}_\alpha^{p*} = \mathbf{S}_\alpha^{\mathbf{x}}) \wedge (\mathbf{S}_\alpha^{p*} = (-1)\cdot \mathbf{S}_\alpha^{\mathbf{x}}) \quad (74)$$

then extreme solution can be calculated using the following formulas

$$u_{\alpha i}^-(\mathbf{x}) = N_{ij}(\mathbf{x},\mathbf{h}^{lower}(\mathbf{S}_\alpha^{p*},\hat{\mathbf{h}}_\alpha))u_{\alpha j}^{p*-} \quad (75)$$

$$u_{\alpha i}^+(\mathbf{x}) = N_{ij}(\mathbf{x},\mathbf{h}^{upper}(\mathbf{S}_\alpha^{p*},\hat{\mathbf{h}}_\alpha))u_{\alpha j}^{p*+} \quad (76)$$

$$\hat{u}_{\alpha i}(\mathbf{x}) = [\hat{u}_{\alpha i}^-(\mathbf{x}), \hat{u}_{\alpha i}^+(\mathbf{x})] \quad (77)$$

If the sign vector $\mathbf{S}_\alpha^{\mathbf{x}}$ is unique, then we have to calculate a new interval solution

$$\hat{\mathbf{u}}_{\alpha}^{k+1*} = [\mathbf{u}\left(\mathbf{h}^{lower}(\mathbf{S}_{\alpha}^{\mathbf{x}}, \hat{\mathbf{h}})\right), \mathbf{u}\left(\mathbf{h}^{upper}(\mathbf{S}_{\alpha}^{\mathbf{x}}, \hat{\mathbf{h}})\right)] \quad (78)$$

next

$$\mathbf{S}_{\alpha}^{k+1*} = \mathbf{S}_{\alpha}^{\mathbf{x}} \quad (79)$$

$$IndSign_{\alpha} := IndSign_{\alpha} \cup \{\mathbf{S}_{\alpha}^{k+1*}\} \quad (80)$$

$$IndSoluton_{\alpha} := IndSoution_{\alpha} \cup \{\hat{\mathbf{u}}_{\alpha}^{k+1*}\} \quad (81)$$

Extreme solution can be calculated using the following formulas

$$u_{\alpha i}^{-}(\mathbf{x}) = N_{ij}(\mathbf{x}, \mathbf{h}^{lower}(\mathbf{S}_{\alpha}^{k+1*}, \hat{\mathbf{h}}_{\alpha}))u_{\alpha j}^{k+1*-} \quad (82)$$

$$u_{\alpha i}^{+}(\mathbf{x}) = N_{ij}(\mathbf{x}, \mathbf{h}^{upper}(\mathbf{S}_{\alpha}^{k+1*}, \hat{\mathbf{h}}_{\alpha}))u_{\alpha j}^{k+1*+} \quad (83)$$

$$\hat{u}_{\alpha i}(\mathbf{x}) = [\hat{u}_{\alpha i}^{-}(\mathbf{x}), \hat{u}_{\alpha i}^{+}(\mathbf{x})] \quad (84)$$

7 Calculation of the value of fuzzy function

In technical applications very often we have to calculate the value of function, which depends on the solution of fuzzy partial differential equations e.g. $y = f(\mathbf{x}, \mathbf{u}, \mathbf{h})$.

Extreme values of function f can be calculated using sensitivity analysis:

$$\mathbf{S}_{\alpha}^{\mathbf{x}} = \left[sign\left(\frac{\partial f(\mathbf{x}, \mathbf{u}(\mathbf{h}_0), \mathbf{h}_0)}{\partial h_1}\right), \ldots, sign\left(\frac{\partial f(\mathbf{x}, \mathbf{u}(\mathbf{h}_0), \mathbf{h}_0)}{\partial h_m}\right) \right] \quad (85)$$

where

$$\frac{\partial f(\mathbf{x},\mathbf{u}(\mathbf{h}_0),\mathbf{h}_0)}{\partial h_i} = \frac{\partial f(\mathbf{x},\mathbf{u}(\mathbf{h}_0),\mathbf{h}_0)}{\partial \mathbf{u}} \frac{\partial \mathbf{u}(\mathbf{h}_0)}{\partial h_i} + \frac{\partial f(\mathbf{x},\mathbf{u}(\mathbf{h}_0),\mathbf{h}_0)}{\partial h_i} \quad (86)$$

Now we can apply similar procedure like in previous paragraph.
If the sign vector $\mathbf{S}_\alpha^\mathbf{x}$ is not unique, then extreme solution can be calculated using the following formulas

$$f_\alpha^-(\mathbf{x}) = f(\mathbf{x}, \mathbf{u}_\alpha^{p*-}, \mathbf{h}^{lower}(\mathbf{S}_\alpha^{p*}, \hat{\mathbf{h}}_\alpha)) \quad (87)$$

$$f_\alpha^+(\mathbf{x}) = f(\mathbf{x}, \mathbf{u}_\alpha^{p*+}, \mathbf{h}^{upper}(\mathbf{S}_\alpha^{p*}, \hat{\mathbf{h}}_\alpha)) \quad (88)$$

If the sign vector $\mathbf{S}_\alpha^\mathbf{x}$ is unique, then calculate a new interval solution

$$\hat{\mathbf{u}}_\alpha^{k+1*} = [\mathbf{u}(\mathbf{h}^{lower}(\mathbf{S}_\alpha^\mathbf{x}, \hat{\mathbf{h}})), \mathbf{u}(\mathbf{h}^{upper}(\mathbf{S}_\alpha^\mathbf{x}, \hat{\mathbf{h}}))] \quad (89)$$

next

$$\mathbf{S}_\alpha^{k+1*} = \mathbf{S}_\alpha^\mathbf{x} \quad (90)$$

$$IndSign_\alpha := IndSign_\alpha \cup \{\mathbf{S}_\alpha^{k+1*}\} \quad (91)$$

$$IndSoluton_\alpha := IndSoution_\alpha \cup \{\hat{\mathbf{u}}_\alpha^{k+1*}\} \quad (92)$$

Extreme values of the function f can be calculated using the following formulas

$$f_\alpha^-(\mathbf{x}) = f(\mathbf{x}, \mathbf{u}_\alpha^{k+1*-}, \mathbf{h}^{lower}(\mathbf{S}_\alpha^{k+1*}, \hat{\mathbf{h}}_\alpha)) \quad (93)$$

$$f_\alpha^+(\mathbf{x}) = f(\mathbf{x}, \mathbf{u}_\alpha^{k+1*+}, \mathbf{h}^{upper}(\mathbf{S}_\alpha^{k+1*}, \hat{\mathbf{h}}_\alpha)) \quad (94)$$

8 Numerical example – plane stress problem in theory of elasticity

Let us consider the following partial differential equations

$$\frac{E}{2(1+\nu)}u_{\alpha,\beta\beta}+\frac{E}{2(1-\nu)}u_{\beta,\beta\alpha}+\rho f_{\alpha}=0,\quad \alpha,\beta=1,2$$
$$u_{\alpha}=\overset{*}{u}_{\alpha},\quad x\in\partial\Omega_{u} \tag{95}$$
$$\sigma_{\alpha\beta}n_{\beta}=\overset{*}{t}_{\alpha},\quad x\in\partial\Omega_{\sigma}$$

where E is a module of elasticity, ν is a Poisson's ratio, u_{α} are displacements, ρ is a mass density, f_{α} are mass forces, $\sigma_{\alpha\beta}$ are stress, n_{α} coordinate of the unit vector which is normal to the boundary $\partial\Omega$, $\overset{*}{t}_{\alpha}$ are boundary traction.
These are equilibrium equations of the plane stress elasticity problem. We can write these equations in the variational form

$$\int_{\Omega}\sigma_{\mathrm{ij}}\delta\varepsilon_{ij}dV=\int_{\Omega}\rho f_{i}\delta u_{i}d\Omega+\int_{\partial\Omega}t_{i}\delta u_{i}dS \tag{96}$$

where ε_{ij} is a strain tensor. If we take into account the constitutive equations

$$\sigma_{ij}=C_{ijkl}\varepsilon_{kl} \tag{97}$$

and geometric equations

$$\varepsilon_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i}) \tag{98}$$

we can define the bilinear form

$$a(\mathbf{u},\mathbf{v}) = \int_{\Omega} \sigma_{ij} \delta\varepsilon_{ij} \, d\Omega = \int_{\Omega} C_{ijkl} u_{i,j} v_{k,l} \, d\Omega \tag{99}$$

and the linear form

$$l(\mathbf{v}) = \int_{\Omega} \rho f_i v_i \, d\Omega + \int_{\partial\Omega} t_i v_i \, d\Omega \tag{100}$$

The variational equations of the theory of elasticity can be written in the following form

$$\forall \mathbf{v} \in V, \quad a(\mathbf{u},\mathbf{v}) = l(\mathbf{v}) \tag{101}$$

Now we can solve this equations using FEM method. The local stiffness matrix can be written in the following form

$$\mathbf{K}^e = \int_{\Omega^e} \mathbf{B}^{e^T} \mathbf{D}^e \mathbf{B}^e \, d\Omega \tag{102}$$

where

$$\mathbf{B}^e = \begin{bmatrix} \frac{\partial N_1^e}{\partial x_1} & 0 & \frac{\partial N_2^e}{\partial x_1} & 0 & \frac{\partial N_3^e}{\partial x_1} & 0 \\ 0 & \frac{\partial N_1^e}{\partial x_2} & 0 & \frac{\partial N_2^e}{\partial x_2} & 0 & \frac{\partial N_3^e}{\partial x_2} \\ \frac{\partial N_1^e}{\partial x_1} & \frac{\partial N_1^e}{\partial x_2} & \frac{\partial N_2^e}{\partial x_2} & \frac{\partial N_2^e}{\partial x_1} & \frac{\partial N_3^e}{\partial x_2} & \frac{\partial N_3^e}{\partial x_1} \end{bmatrix} \tag{103}$$

$$\mathbf{D}^e = \frac{E^e}{1-{\nu^e}^2} \begin{bmatrix} 1 & \nu^e & 0 \\ \nu^e & 1 & 0 \\ 0 & 0 & \frac{1-\nu^e}{2} \end{bmatrix} \tag{104}$$

and N_i are shape functions, which will be described later.
The load vector can be calculated from the following equations:

$$\mathbf{Q} = \int_{\Omega} \mathbf{N}^T \rho \mathbf{f} d\Omega + \int_{\partial\Omega} \mathbf{N}^T \mathbf{t} dS \tag{105}$$

In calculation, we will be use triangular element, which is shown in Fig. 1.

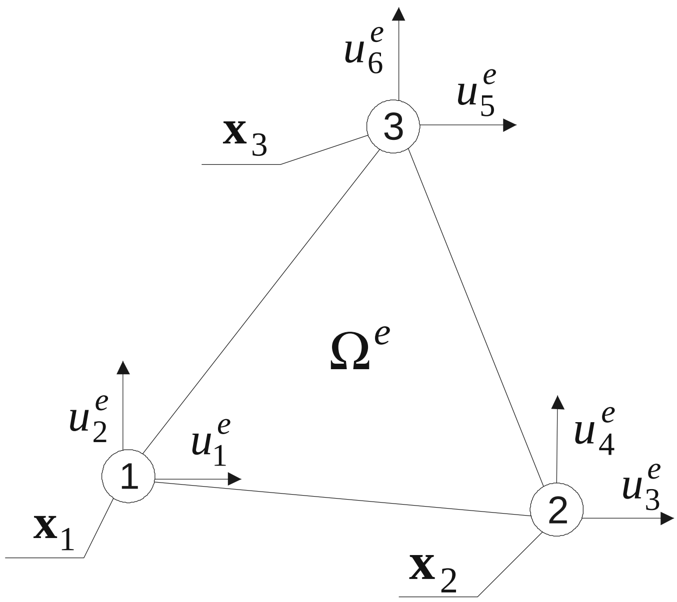

Fig. 1

Displacement in the element can be described in the following way:

$$\mathbf{u}^e(x) = \begin{bmatrix} u_1^e(x) \\ u_2^e(x) \end{bmatrix} = \begin{bmatrix} N_1^e(\mathbf{x}) & 0 \\ 0 & N_1^e(\mathbf{x}) \\ N_2^e(\mathbf{x}) & 0 \\ 0 & N_2^e(\mathbf{x}) \\ N_3^e(\mathbf{x}) & 0 \\ 0 & N_3^e(\mathbf{x}) \end{bmatrix}^T \cdot \begin{bmatrix} u_1^e \\ u_2^e \\ u_3^e \\ u_4^e \\ u_5^e \\ u_6^e \end{bmatrix} = \mathbf{N}^e(\mathbf{x})\mathbf{u}^e \quad (106)$$

Shape functions of the element “e” satisfy the following conditions:

$$N_i^e(\mathbf{x}_j) = \delta_{ij} \quad (107)$$

If we assume, that the shape functions are linear i.e.

$$N_i^e(\mathbf{x}) = a_i^e + b_i^e x_1 + c_i^e x_2 \quad (108)$$

then the function $N_1^e(\mathbf{x})$ has the following form:

$$N_1^e(\mathbf{x}) = \frac{x_1^{2e} x_2^{3e} - x_1^{3e} x_2^{2e} + (x_2^{2e} - x_2^{3e}) x_1 + (x_1^{3e} - x_1^{2e}) x_2}{\Delta^e} \quad (109)$$

where

$$\Delta^e = \begin{vmatrix} 1 & x_1^{1e} & x_2^{1e} \\ 1 & x_1^{2e} & x_2^{2e} \\ 1 & x_1^{3e} & x_2^{3e} \end{vmatrix} \quad (110)$$

etc.

Let us consider structure, which is shown in Fig. 2.

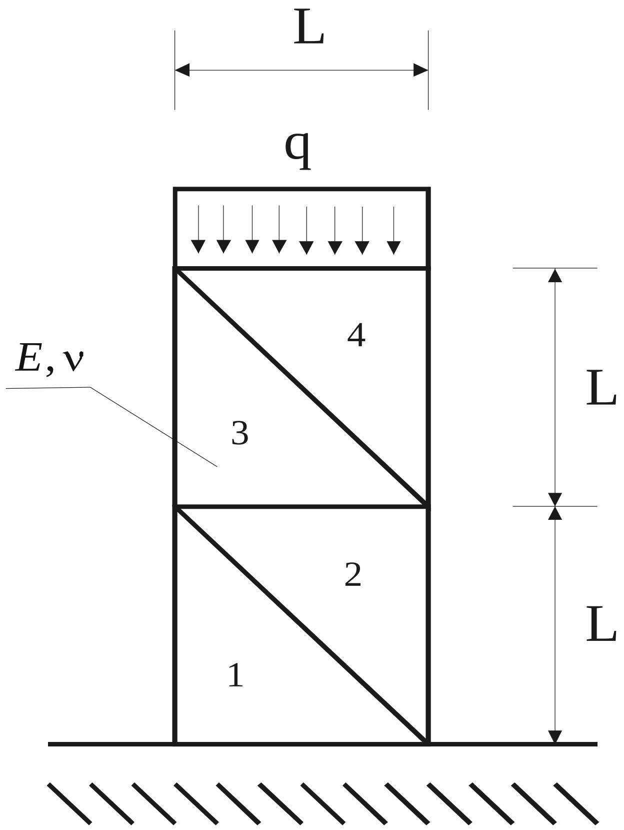

Fig. 2

In calculation we assume that L=1 [m], $q = 1\left[\frac{kN}{m}\right]$, $\nu = 0.3$.

Table 1. Fuzzy Young's modulus

α	α=0	α=1
$\hat{E}_{\alpha}^{1}$	[189, 231][GPa]	210 [GPa]
$\hat{E}_{\alpha}^{2}$	[189, 231][GPa]	210 [GPa]
$\hat{E}_{\alpha}^{3}$	[189, 231] [GPa]	210 [GPa]
$\hat{E}_{\alpha}^{4}$	[189, 231] [GPa]	210 [GPa]

After applying algorithm 3 we get the following numerical results:

Table 2. Fuzzy stress

α	α=0	α=1
$\hat{\sigma}_{y\alpha}^{1}$	[0.96749, 0.974493] [kPa]	0.971063 [kPa]
$\hat{\sigma}_{y\alpha}^{2}$	[1.02833, 1.02955] [kPa]	1.02894 [kPa]
$\hat{\sigma}_{y\alpha}^{3}$	[0.98086, 1.01719] [kPa]	0.999086 [kPa]
$\hat{\sigma}_{y\alpha}^{4}$	[0.982807, 1.01914] [kPa]	1.00091 [kPa]

Table 3. Fuzzy displacements

No.	$\hat{u}_{i\alpha}, \alpha = 0$ [m]
1	[0, 0]
2	[0, 0]
3	[0, 0]
4	[0, 0]
5	[3.2517e-14,7.49058e-13]
6	[3.81132e-12, 4.692e-12]
7	[-1.5243e-12,-4.9879e-13]
8	[4.4199e-12, 5.4275e-12]
9	[-1.5134e-12,1.0498e-12]
10	[8.1381e-12,9.9465e-12]
11	[-3.1758e-12,-1.7949e-13]
12	[8.7620e-12,1.0709e-11]

This problem has 8 degree of freedom.

9 Numerical example – plane stress problem in theory of elasticity

The equilibrium equations of a rod has the following form

$$\begin{cases} \frac{d}{dx}\left(EA\frac{du}{dx}\right)+n=0 \\ u\in V \end{cases}. \tag{111}$$

where E is an Young's module, A is an area of cross section, n is a load and V is some functional space. We can formulate the problem (111) in the variational form:

$$\forall v\in V,\quad a(u,v)=l(v) \tag{112}$$

where

$$a(u,v)=\int_0^L EA\frac{du}{dx}\frac{dv}{dx}dx \tag{113}$$

$$l(v)=\int_0^L nvdx+... \tag{114}$$

To solution of the problem (112) we can apply finite element method. Local stiffness matrix (in local coordinate system) has the following form:

$$\overline{\mathbf{K}}^e = \int_{V^e}(\mathbf{B}^e)^T\mathbf{D}^e\mathbf{B}^e dx = \frac{E^e A^e}{L^e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{115}$$

where

$$\mathbf{B}^e = \left[-\frac{1}{L^e}, \frac{1}{L^e} \right] \tag{116}$$

$$\mathbf{D}^e = [E^e] \tag{117}$$

Local stiffness matrix in global coordinate system has the following form:

$$\mathbf{K}^e = \int_{V^e} (\mathbf{C}^e)^T (\mathbf{B}^e)^T \mathbf{D}^e \mathbf{B}^e \mathbf{C}^e dx \tag{118}$$

where rotation matrix has the following form

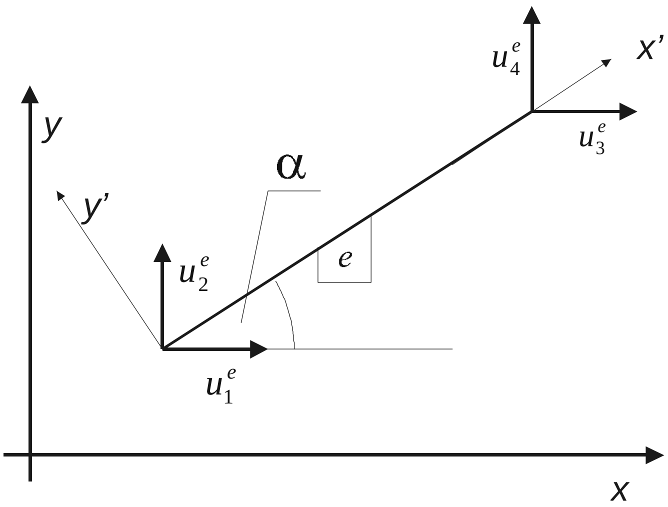

Fig. 3

$$\mathbf{C}^e = \begin{bmatrix} cos(\alpha^e) & sin(\alpha^e) & 0 & 0 \\ -sin(\alpha^e) & cos(\alpha^e) & 0 & 0 \\ 0 & 0 & cos(\alpha^e) & sin(\alpha^e) \\ 0 & 0 & -sin(\alpha^e) & cos(\alpha^e) \end{bmatrix} \tag{119}$$

Local load vector in global coordinate system:

$$\mathbf{Q}^e = \int_{V^e} (\mathbf{C}^e)^T (\mathbf{N}^e)^T \mathbf{n} dx \tag{120}$$

The structure is shown in the Fig. 4.

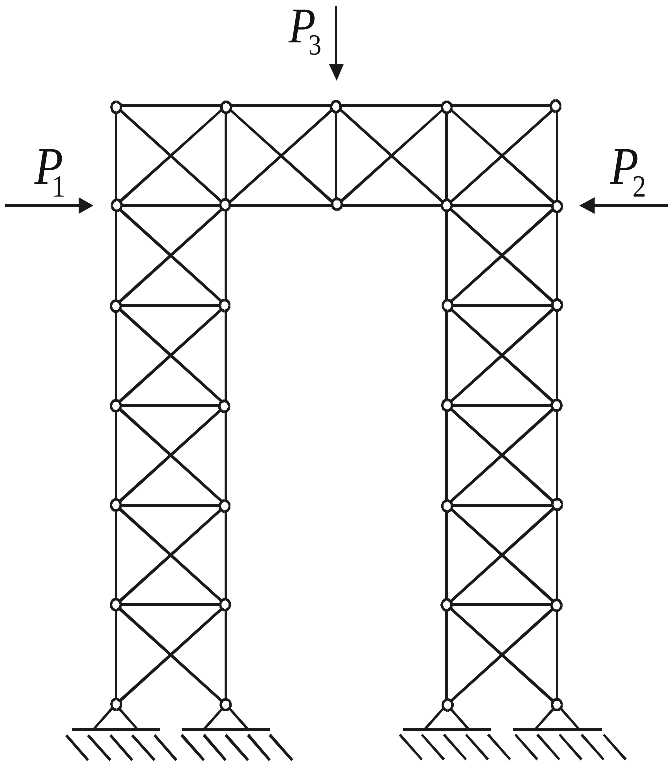

Fig. 4

Numerical data are as follows P=10 [kN], L=1 [m], $\nu = 0.3$, the Young's modulus is the same like in the previous example. Interval axial forces are shown in table 4.

Table 4. Interval axial force

No.	Axial force [*N*]
1	[3145.34, 4393.45]
2	[1482.48, 1914.16]
3	[-172.138, -221.845]
4	[164.454, 279.737]
5	[-958.619, -936.417]
6	[2459.35, 2536.53]
7	[1527.83, 1546.14]
8	[-343.544, -357.966]
9	[1708.72, 1617.27]
10	[-840.883, -841.035]
11	[1132.62, 1189.25]
12	[1532.73, 1547.37]
13	[-338.641, -356.736]
14	[3028.51, 2962.81]
15	[-932.071, -929.76]
16	[-278.358, -245.009]
17	[1656.79, 1671.62]
18	[-214.586, -232.489]
19	[4264.06, 4221.36]
20	[-169.222, -168.335]
21	[-751.05, -742.133]
22	[453.902, 470.55]
23	[-1417.47, -1433.55]
24	[6437.89, 6417.04]
25	[-7444.75, -7432.58]
26	[-200.408, -202.065]
27	[-2196.2, -2197.33]
28	[283.42, 285.763]
29	[4020.01, 4013.59]
30	[-200.408, -202.065]
31	[-9461.8, -9431.91]
32	[3589.87, 3583.79]
33	[-3488.96, -3478.74]
34	[713.715, 704.035]

35	[4929.89, 4924.37]
36	[720.439, 696.638]
37	[3580.36, 3594.25]
38	[-3482.95, -3485.36]
39	[-9466.06, -9427.23]
40	[4010.55, 4024]
41	[-194.644, -208.406]
42	[-2188.83, -2205.43]
43	[275.268, 294.73]
44	[-7448.38, -7428.59]
45	[-194.644, -208.406]
46	[6417.52, 6439.45]
47	[451.658, 473.02]
48	[-1419.72, -1431.08]
49	[-738.486, -755.954]
50	[-166.773, -171.028]
51	[4242.96, 4244.56]
52	[1655.57, 1672.95]
53	[-215.805, -231.149]
54	[-266.518, -258.031]
55	[-930.146, -931.887]
56	[3007.62, 2985.78]
57	[1531.23, 1549.04]
58	[-340.144, -355.068]
59	[1144.66, 1176]
60	[-839.969, -841.95]
61	[1686.62, 1641.68]
62	[1528.04, 1545.77]
63	[-343.334, -358.339]
64	[2470.18, 2524.72]
65	[-947.416, -949.597]
66	[253.654, 185.319]
67	[1683.18, 1701.27]
68	[-188.192, -202.832]
69	[3683.74, 3761.16]

10 Quasi-analytical method

Let us consider boundary value problem with fuzzy parameters

$$\mathbf{L}(\mathbf{x},\mathbf{u},\mathbf{h}) = \mathbf{f}(\mathbf{x},\mathbf{h}), \quad \mathbf{u} \in V, \quad \mathbf{h} \in F \qquad (121)$$

If we differentiate the boundary value problem with respect to h_i we get a new boundary value problem

$$\mathbf{H}\left(\mathbf{x},\mathbf{u},\mathbf{h},\frac{\partial \mathbf{u}}{\partial h_i}\right) = \frac{\partial \mathbf{f}(\mathbf{x},\mathbf{h})}{\partial h_i}, \quad \mathbf{u} \in V, \quad \mathbf{h} \in F \qquad (122)$$

Let us assume that we know the solution of the boundary value problem (121) for $\mathbf{h}_0^{\alpha} = mid(\hat{\mathbf{h}}_{\alpha})$ i.e. $\mathbf{u}_0^{\alpha}(\mathbf{x}) = \mathbf{u}(\mathbf{x},\mathbf{h}_0^{\alpha})$.

Now we can substitute $\mathbf{v}_i = \dfrac{\partial \mathbf{u}}{\partial h_i}$ and we get a new boundary value problem

$$\mathbf{H}\left(\mathbf{x},\mathbf{u}_{\alpha}^{0}(\mathbf{x}),\mathbf{h}_{\alpha}^{0},\mathbf{v}_i\right) = \frac{\partial \mathbf{f}(\mathbf{x},\mathbf{h}_{\alpha}^{0})}{\partial h_i}, \quad \mathbf{v}_i \in V_i \qquad (123)$$

Problem (123) can be solved using any method. If we know the solution of the equations (123) we can calculate extreme solution by using the algorithm 3.

Let us consider the following example

$$\frac{du}{dx} = hx, \; u(0) = u_0 \cdot h^2 \qquad (124)$$

The exact solution is the following

$$u(x,h) = \frac{hx^2}{2} + u_0 \cdot h^2 \qquad (125)$$

If we calculate derivative of the boundary value problem (124) with respect to *h* we get

$$\frac{d}{dx}\left(\frac{\partial u}{\partial h}\right) = x, \ \frac{\partial u(0)}{\partial h} = 2u_0 h \tag{126}$$

or

$$\frac{dv}{dx} = x \cdot h, \ v(0,h) = 2u_0 h \tag{127}$$

where

$$v(x,h) = \frac{\partial u(x,h)}{\partial h} \tag{128}$$

the solution of the equation (127) is the following

$$v(x,h) = \frac{x^2}{2} + 2 \cdot u_0 \cdot h \tag{129}$$

If $h > 0$, then $v(x,h)>0$ and the function $u = u(x,h)$ is monotone (for fixed *x*) and extreme solution can be calculated in the following way:

$$\hat{u}_\alpha(x) = [u_\alpha^-(x), u_\alpha^+(x)] \tag{130}$$

where

$$u_\alpha^-(x) = u(x, h_\alpha^-) = \frac{h_\alpha^- x^2}{2} + u_0 \cdot (h_\alpha^-)^2 \tag{131}$$

$$u_\alpha^+(x) = u(x, h_\alpha^+) = \frac{h_\alpha^+ x^2}{2} + u_0 \cdot (h_\alpha^+)^2 \tag{132}$$

It should be noted that if we know the numerical solution $\mathbf{u}_{\alpha}^{0}$ of the problem (121) we could not calculate $\frac{\partial \mathbf{u}}{\partial h_i}$ directly.

11 Finite difference method

The derivative $\frac{\partial u_i}{\partial h_j}$ can be calculated directly using the finite difference

$$\frac{\partial u_i(\mathbf{x},\mathbf{h}_{\alpha}^{0})}{\partial h_j} \approx \approx \frac{u_i(\mathbf{x},h_{1\alpha}^{0},\ldots,h_{j\alpha}^{0}+\Delta h_j,\ldots,h_{1\alpha}^{0})-u_i(\mathbf{x},h_{1\alpha}^{0},\ldots,h_{j\alpha}^{0},\ldots,h_{1\alpha}^{0})}{\Delta h_j} \tag{133}$$

The value $u_i(\mathbf{x},\mathbf{h}_{\alpha}^{0})$ is a solution of the boundary value problem

$$\mathbf{L}(\mathbf{x},\mathbf{u},\mathbf{h}_{\alpha}^{0}) = \mathbf{f}(\mathbf{x},\mathbf{h}_{\alpha}^{0}), \quad \mathbf{u} \in V. \tag{134}$$

Other methods of calculation of the sensitivity can be find in the book (Kleiber 1997).

12 Point monotonicity tests

12.1 First order monotonicity tests

If derivative of the function $u = u(\mathbf{h})$ has constant sign, then we can assume that the function $u = u(\mathbf{h})$ is monotone. The value of the function $\frac{\partial u}{\partial h_i}$ can be approximated by using the linear function:

$$\frac{\partial u_{(1)}(\mathbf{h})}{\partial h_i} = \frac{\partial u(\mathbf{h}_0)}{\partial h_i} + \sum_{j=1}^{m} \frac{\partial^2 u(\mathbf{h}_0)}{\partial h_i \partial h_j}(h_j - h_j^0) \qquad (135)$$

An interval function is an interval-valued function of one or more interval arguments. Consider a real-valued function *f* of real variables $x_1,...,x_n$ and an interval function $\hat{f}$ of interval variables $\hat{x}_1,...,\hat{x}_n$. The interval function $\hat{f}$ is said to be an interval extension of *f* if

$$\forall (x_1,...,x_n) \in D_f, \quad \hat{f}(x_1,...,x_n) = f(x_1,...,x_n) \qquad (136)$$

where D_f is a domain of the function *f*. That is, if the arguments of $\hat{f}$ are degenerate intervals (i.e. $\hat{x}_i = x_i$), then $\hat{f}(\hat{x}_1,...,\hat{x}_n)$ is a degenerate interval equal to $f(x_1,...,x_n)$.

From properties of interval extensions (Neumaier 1990) arise that

$$\text{if } 0 \notin \frac{\partial \hat{u}_{(1)}(\hat{\mathbf{h}}_\alpha)}{\partial h_i}, \text{ then } \forall \mathbf{h} \in \hat{\mathbf{h}}_\alpha, \ \frac{\partial u_{(1)}(\mathbf{h})}{\partial h_i} \neq 0. \qquad (137)$$

and we can assume that the function $u = u(\mathbf{h})$ is monotone.

12.2 High order monotonicity tests

We can also approximate derivative of the function $u = u(\mathbf{h})$ using high order polynomials

$$\frac{\partial u_{(2)}(\mathbf{h})}{\partial h_i} = \frac{\partial u(\mathbf{h}_0)}{\partial h_i} + \sum_{j=1}^{m} \frac{\partial^2 u(\mathbf{h}_0)}{\partial h_i \partial h_j}(h_j - h_j^0) + \\ + \frac{1}{2}\sum_{j=1}^{m}\sum_{k=1}^{m} \frac{\partial^3 u(\mathbf{h}_0)}{\partial h_i \partial h_j \partial h_k}(h_j - h_j^0)(h_k - h_k^0) \tag{138}$$

$$\text{If } 0 \notin \frac{\partial \hat{u}_{(p)}(\hat{\mathbf{h}}_\alpha)}{\partial h_i}, \text{ then } \forall \mathbf{h} \in \hat{\mathbf{h}}_\alpha, \ \frac{\partial u_{(p)}(\mathbf{h})}{\partial h_i} \neq 0\,. \tag{139}$$

and we can assume that the function $u = u(\mathbf{h})$ is monotone.

13 Numerical example - displacement of the shell structure

The equilibrium equations of shell structures can be written in the following form:

$$\begin{aligned} & T^{\beta\alpha}|_\beta - b_\gamma^\alpha M^{\beta\gamma}|_\beta + b^\alpha = 0 \\ & T^{\beta\alpha} b_{\alpha\beta} + M^{\alpha\beta}|_{\beta\beta} + b^3 = 0 \\ & T^{\beta\alpha} n_\beta + b_\gamma^\alpha M^{\beta\gamma} n_\beta = p^\alpha, \quad x \in \partial\Omega \\ & M^{\beta\alpha}|_\beta n_\alpha + \frac{d}{ds}(M^{\alpha\beta}\tau_\alpha n_\beta) = p^3, \quad x \in \partial\Omega \end{aligned} \tag{140}$$

where

$$u^{\alpha}|_{\beta}=u^{\alpha},_{\beta}+\begin{Bmatrix}\alpha\\ \beta\gamma\end{Bmatrix}u^{\gamma},\quad u^{\alpha},_{\beta}=\frac{\partial u^{\alpha}}{\partial x^{\beta}},\quad \alpha,\beta=1,2 \quad (141)$$

$$\begin{Bmatrix} i\\ jk\end{Bmatrix}=\frac{1}{2}g^{il}(g_{jl,k}+g_{kl.j}-g_{jk.l}) \quad (142)$$

g_{ij} is a metric tensor.

Let us consider shell structure, which is shown in Fig. 5.

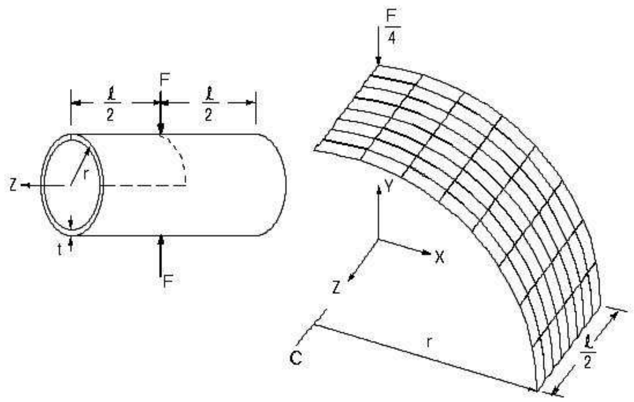

Fig. 5

In calculation we assume the following numerical data $E\in[2.0\cdot10^{5},2.2\cdot10^{5}][MPa]$, $\nu\in[0.2,0.3]$, L=0.263 [m], r=0.126 [m], F=444.8 [N], $t=2.38\cdot10^{-3}[m]$. We will be looking for an interval displacement in direction of the force F. Usign first order monotonicity test we can check monotonicity of the solution. Because the function $u=u(E,\nu)$ is monotone, then extreme values of the solution can be found using only the endpoints of given intervals. The interval solution is as follows:

$$\alpha=0:\ u\in[-0.043514,-0.03748]\,[m], \quad (143)$$

$$\alpha=1:\ u = -\ 0.04102\ [m]. \tag{144}$$

Using point monotonicity test we can calculate the interval solution only in some selected points. In this example professional FEM program Ansys was applied.

14 Taylor model of the solution

If the solution is sufficiently smooth, then we can approximate them by using Taylor series

$$u_{i\alpha}(\mathbf{h}) = u_i(\mathbf{h}_\alpha^0) + \sum_{j=1}^{m} \frac{\partial u_i(\mathbf{h}_\alpha^0)}{\partial h_j}(h_i - h_\alpha^0)\,. \tag{145}$$

Extreme values of the solution can be approximated directly by using equation (145) and interval arithmetic

$$\hat{u}_{i\alpha} = \hat{u}_{i\alpha}(\hat{\mathbf{h}}_\alpha) = u_i(\mathbf{h}_\alpha^0) + \sum_{j=1}^{m} \frac{\partial u_i(\mathbf{h}_\alpha^0)}{\partial h_j}(\hat{h}_{i\alpha} - h_\alpha^0)\,. \tag{146}$$

This method has very low computational complexity (m+1 system of equations) (Akapan et all 2001). Unfortunately, the equation (146) gives only approximate solution.

15 Interval monotonicity tests

15.1 Linear equations

Let us consider the problem (36) and assume that we know the solution of the following systems of linear interval equation

$$\hat{\mathbf{K}}(\hat{\mathbf{h}}_{\alpha})\mathbf{u} = \hat{\mathbf{Q}}(\hat{\mathbf{h}}_{\alpha})\,. \tag{147}$$

$$\hat{\mathbf{K}}(\hat{\mathbf{h}}_{\alpha})\frac{\partial \mathbf{u}}{\partial h_i} = \frac{\partial \hat{\mathbf{Q}}(\hat{\mathbf{h}}_{\alpha})}{\partial h_i} - \frac{\partial \hat{\mathbf{K}}(\hat{\mathbf{h}}_{\alpha})}{\partial h_i}\hat{\mathbf{u}}(\hat{\mathbf{h}}_{\alpha})\,. \tag{148}$$

where

$$\hat{\mathbf{u}}(\hat{\mathbf{h}}_{\alpha}) = hull\sum(\hat{\mathbf{K}}(\hat{\mathbf{h}}_{\alpha}),\hat{\mathbf{Q}}(\hat{\mathbf{h}}_{\alpha}))\,. \tag{149}$$

If

$$0 \notin \frac{\partial \hat{u}_i(\hat{\mathbf{h}}_{\alpha})}{\partial h_j} = hull\sum\left(\hat{\mathbf{K}}(\hat{\mathbf{h}}_{\alpha}),\frac{\partial \hat{\mathbf{Q}}(\hat{\mathbf{h}}_{\alpha})}{\partial h_j} - \frac{\partial \hat{\mathbf{K}}(\hat{\mathbf{h}}_{\alpha})}{\partial h_j}\hat{\mathbf{u}}_{\alpha}(\hat{\mathbf{h}}_{\alpha})\right), \tag{150}$$

then the solution of the problem (36) is monotone (with guaranteed accuracy).

15.2 Numerical example – heat transfer

Let's us consider heat transfer problem

$$\begin{cases} R_1 < r < R_2 : & \dfrac{1}{r}\dfrac{d}{dr}\left(r\lambda \dfrac{dT(r)}{dr}\right) + Q = 0 \\ r = R_1 & : \ -\lambda \dfrac{dT(r)}{dr} = \alpha\left(T(r) - T_b\right) \\ r = R_2 & : \ T(r) = T_t \end{cases} \quad . \tag{151}$$

In calculation we assume the following numerical data R_1=0.0005 $[m]$, $R_2 = 10 \cdot R_1$, $\alpha = 2000\left[\dfrac{W}{m^2 \cdot K}\right]$, $T_b = 32^o C$, $T_t = 37^o C$, Q= 10245 $\left[\dfrac{W}{m^3}\right]$, $\lambda \in [0.21, 0.23]\left[\dfrac{W}{m \cdot K}\right]$.

Numerical solutions are shown in the table 5.

Table 5. Interval temperature

No	$T_i^-[^oC]$	$T_i^+[^oC]$
1	36.586	36.619
2	35.470	35.494
3	34.782	34.800
4	34.282	34.298
5	33.894	33.582
6	33.302	33.308
7	33.065	33.070
8	32.857	32.859
9	32.669	32.671
10	32.500	32.500

15.3 Nonlinear equations

Sometimes system of algebraic equations is nonlinear

$$\mathbf{F}(\mathbf{x},\mathbf{u},\mathbf{h}) = \mathbf{0}, \quad \mathbf{h} \in \hat{\mathbf{h}}_{\alpha}. \tag{152}$$

From implicit function theorem arise that

$$\frac{\partial \mathbf{F}}{\partial \mathbf{u}}\frac{\partial \mathbf{u}}{\partial h_i} = -\frac{\partial \mathbf{F}}{\partial h_i}, \quad i=1,...,m. \tag{153}$$

Equation (153) is a system of linear equation with unknown $\frac{\partial \mathbf{u}}{\partial h_i}$, because of that

$$\frac{\partial u_i}{\partial h_j} = -\frac{\left\| \begin{bmatrix} \frac{\partial F_1}{\partial u_1} & \cdots & \frac{\partial F_1}{\partial u_{i-1}} & \frac{\partial F_1}{\partial h_j} & \frac{\partial F_1}{\partial u_{i-1}} & \cdots & \frac{\partial F_1}{\partial u_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial F_n}{\partial u_{11}} & \cdots & \frac{\partial F_n}{\partial u_{i-1}} & \frac{\partial F_n}{\partial h_j} & \frac{\partial F_n}{\partial u_{i+1}} & \cdots & \frac{\partial F_n}{\partial u_n} \end{bmatrix} \right\|}{\left| \frac{\partial \mathbf{F}}{\partial \mathbf{u}} \right|}, \tag{154}$$

$$\frac{\partial u_i}{\partial h_j} = -\frac{\left| \frac{\partial \mathbf{F}}{\partial (u_1,...,u_{i-1},h_j,u_{i+1},...,u_n)} \right|}{\left| \frac{\partial \mathbf{F}}{\partial \mathbf{u}} \right|}. \tag{155}$$

From equation (155) arrays, that if the following determinates

$$\left| \frac{\partial \mathbf{F}}{\partial \mathbf{u}} \right|, \quad \left| \frac{\partial \mathbf{F}}{\partial (u_1,...,u_{i-1},h_j,u_{i+1},...,u_n)} \right|. \tag{156}$$

have constant sign, then the derivative $\frac{\partial u_i}{\partial h_j}$ has also constant sign and the functions $u_i = u_i(\ldots,h_j,\ldots)$ are monotone.

From properties of the determinates and Darbox theorem arise, that if

$$\forall \mathbf{h} \in \hat{\mathbf{h}}_\alpha, \left| \frac{\partial \mathbf{F}(\mathbf{x},\mathbf{u}(\mathbf{h}),\mathbf{h})}{\partial(u_1,\ldots,u_{i-1},h_j,u_{i+1},\ldots,u_n)} \right| \neq 0, \ , \qquad (157)$$

$$\forall \mathbf{h} \in \hat{\mathbf{h}}_\alpha, \left| \frac{\partial \mathbf{F}(\mathbf{x},\mathbf{u}(\mathbf{h}),\mathbf{h})}{\partial \mathbf{u}} \right| \neq 0. \ . \qquad (158)$$

i.e. the Jacobean matrix are regular, then the functions $u_i = u_i(\ldots,h_j,\ldots)$ are monotone.

From properties of interval arithmetic, arise that

$$\forall \mathbf{h} \in \hat{\mathbf{h}}_\alpha, \ \left| \frac{\partial \mathbf{F}(\mathbf{x},\mathbf{u}(\mathbf{h}),\mathbf{h})}{\partial \mathbf{u}} \right| \in \left| \frac{\partial \hat{\mathbf{F}}(\mathbf{x},\mathbf{u}(\hat{\mathbf{h}}_\alpha),\hat{\mathbf{h}}_\alpha)}{\partial \mathbf{u}} \right| \qquad (159)$$

and

$$\begin{aligned} &\forall \mathbf{h} \in \hat{\mathbf{h}}_\alpha, \ \left| \frac{\partial \mathbf{F}(\mathbf{x},\mathbf{u}(\mathbf{h}),\mathbf{h})}{\partial(u_1,\ldots,u_{i-1},h_j,u_{i+1},\ldots,u_n)} \right| \in \\ &\in \left| \frac{\partial \hat{\mathbf{F}}(\mathbf{x},\mathbf{u}(\hat{\mathbf{h}}_\alpha),\hat{\mathbf{h}}_\alpha)}{\partial(u_1,\ldots,u_{i-1},h_j,u_{i+1},\ldots,u_n)} \right|. \end{aligned} \ . \qquad (160)$$

We can see that, if the interval Jacobean matrices (159, 160) are regular, then the functions $u_i = u_i(\mathbf{h})$ are monotone.

15.4 Numerical example – frame structure

The equilibrium equations of beam is as follows:

$$\frac{d^2}{dx^2}\left(EJ\frac{d^2u}{dx^2}\right)=q,\quad u\in V. \tag{161}$$

If we apply the finite element, we get equilibrium equations in the following form:

$$\mathbf{K}(\mathbf{h})\mathbf{u}=\mathbf{Q}. \tag{162}$$

Let us consider a structure, which is shown in the Fig. 6.

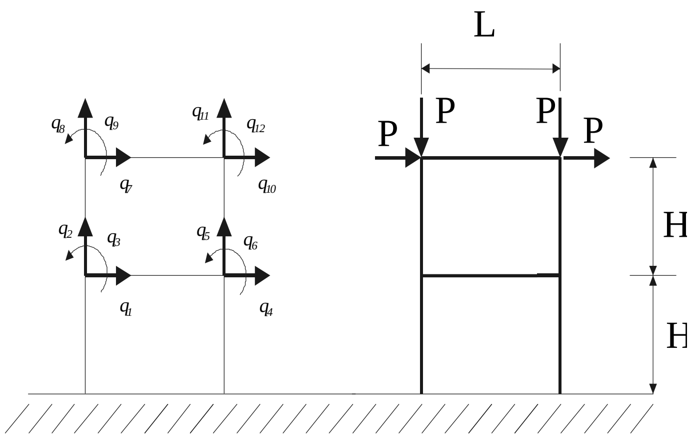

Fig. 6

In calculation we assume the following data $E\in[210,220][GPa]$, $J\in\left[\frac{0.05^4}{12},\frac{0.055^4}{12}\right][m^4]$, $A\in[0.05^2,0.055^2][m^2]$, L=H=1 $[m]$, P=1 $[kN]$.

Table 6. Interval displacements

No.	q_i^- $[m]$	q_i^+ $[m]$
1	0.035716	0.037414
2	0.000008	0.000009
3	-0.011230	-0.010718
4	0.035716	0.037414
5	-0.000021	-0.000017
6	-0.011230	-0.010718
7	0.082163	0.086067
8	0.00009	0.000010
9	-0.007494	-0.007151
10	0.082163	0.086067
11	-0.000033	-0.000026
12	-0.007494	-0.007151

15.5 Subdivision

The interval extension of the Jacobean matrix may become singular even for very narrow intervals $\hat{\mathbf{h}}_{\alpha}$. In this case, we can divide these intervals and repeat procedure again.

16 Optimization methods

16.1 Description of the algorithm

If the intervals $\hat{\mathbf{h}}_\alpha$ are very wide, then we cannot apply methods, which were described below. In such situation, optimization methods can be applied.

$$u_{i\alpha}^- \Leftarrow \begin{cases} \min u_i \\ \mathbf{L}(\mathbf{x},\mathbf{u},\mathbf{h}) = \mathbf{f}(\mathbf{h}) \\ \mathbf{h} \in \hat{\mathbf{h}}_\alpha, \mathbf{u} \in V \end{cases} \qquad u_{i\alpha}^+ \Leftarrow \begin{cases} \max u_i \\ \mathbf{L}(\mathbf{x},\mathbf{u},\mathbf{h}) = \mathbf{f}(\mathbf{h}) \\ \mathbf{h} \in \hat{\mathbf{h}}_\alpha, \mathbf{u} \in V \end{cases}. \quad (163)$$

Approximate solution can be defined as follows:

$$u_{i\alpha}^- \Leftarrow \begin{cases} \min u_i \\ \mathbf{K}(\mathbf{h})\mathbf{u} = \mathbf{Q}(\mathbf{h}) \\ \mathbf{h} \in \hat{\mathbf{h}}_\alpha \end{cases} \qquad u_{i\alpha}^+ \Leftarrow \begin{cases} \max u_i \\ \mathbf{K}(\mathbf{h})\mathbf{u} = \mathbf{Q}(\mathbf{h}) \\ \mathbf{h} \in \hat{\mathbf{h}}_\alpha \end{cases}. \quad (164)$$

16.2 Numerical example – displacements of beam

Let us consider beam structure, which is shown in Fig. 7.

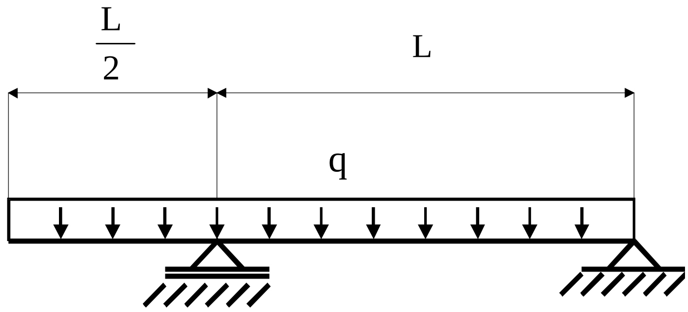

Fig. 7

The equilibrium equation has the following form

$$\begin{cases} \dfrac{d^2}{dx^2}\left(EJ\,\dfrac{d^2u}{dx^2} \right) = q(x), \\ u\left(\dfrac{L}{2}\right) = 0,\ \ u\left(\dfrac{3L}{2}\right) = 0,\ \ \dfrac{d^2u(0)}{dx^2} = 0,\ \ \dfrac{d^2}{dx^2}u\left(\dfrac{3L}{2}\right) = 0 \end{cases} . \quad (165)$$

In calculations we assume that $E \in [2.0 \cdot 10^5, 2.2 \cdot 10^5][MPa]$, $J \in \left[\dfrac{0.049^4}{12}, \dfrac{0.051^4}{12} \right][m^4]$, $L = [0.999, 1.001][m]$, $q \in [9.9, 10.1][kN]$. Numerical results are shown in Fig. 8.

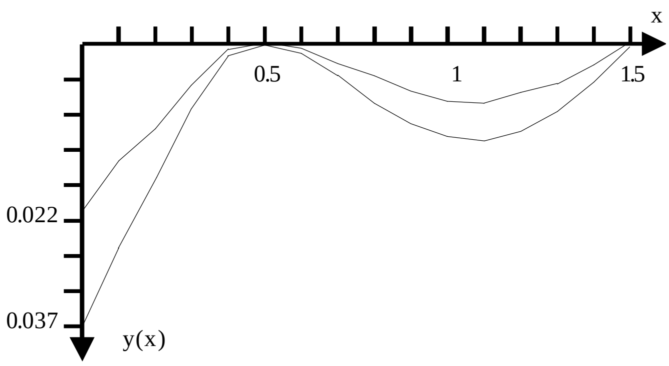

Fig. 8

17 Conclusions

1) Calculation of the solutions of the fuzzy partial differential equations is in general very difficult (NP-hard).
2) In engineering applications the relation between the solution and the uncertain parameters is usually monotone.
3) Using methods which are based on sensitivity analysis we can solve very complicated problems of computational mechanics (even with thousands degree of freedom).
4) If we apply the point monotonicity tests we can use results, which were generated by the existing engineering software.
5) Reliable methods of solution of the fuzzy partial differential equations are based on the interval arithmetic. These methods have high computational complexity.
6) In some cases (e.g. if we know analytical solution) the optimization method can be applied.
7) In some special cases we can predict the solution of the fuzzy partial differential equations.
8) The fuzzy partial differential equation can be applied to modeling of mechanical systems (structures) with uncertain parameters.

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