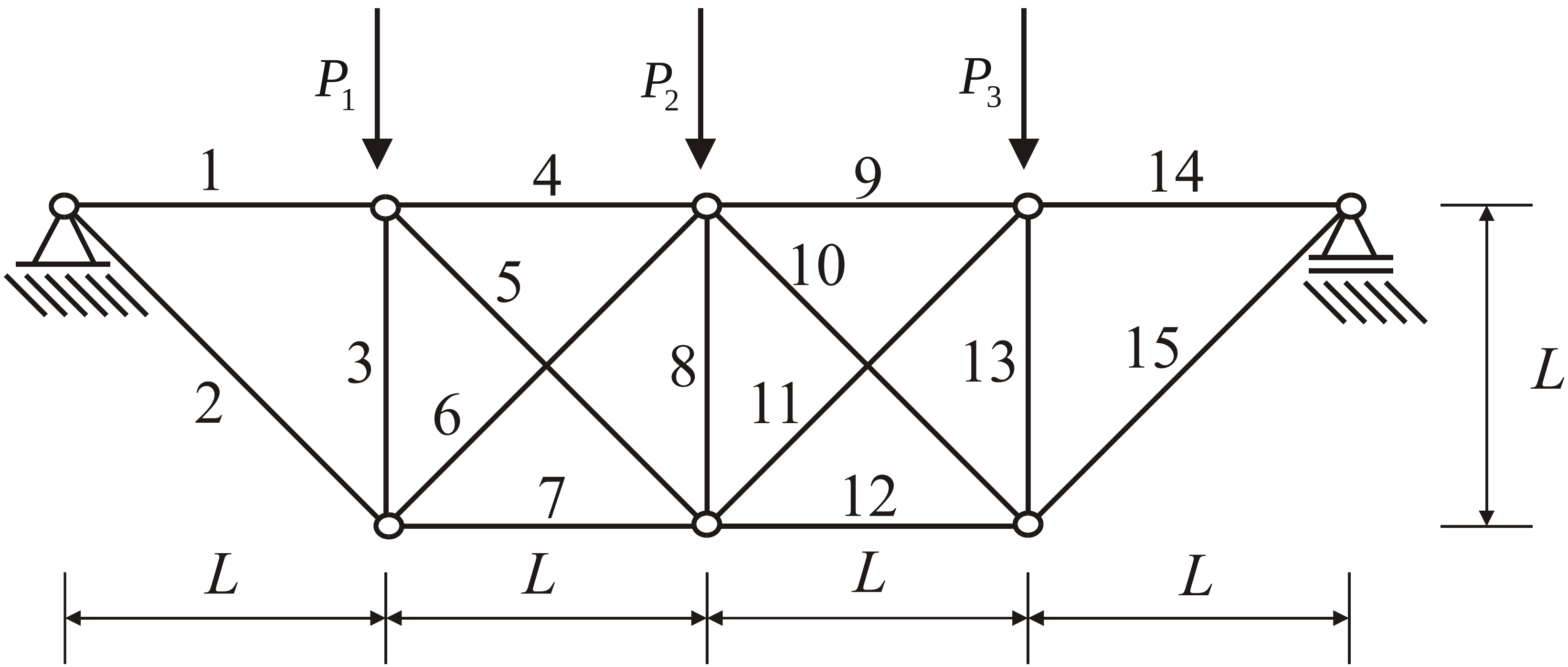
Modeling of Uncertainty in Computational Mechanics

Andrzej Pownuk,
The University of Texas at El Paso
http://andrzej.pownuk.com

Truss structure

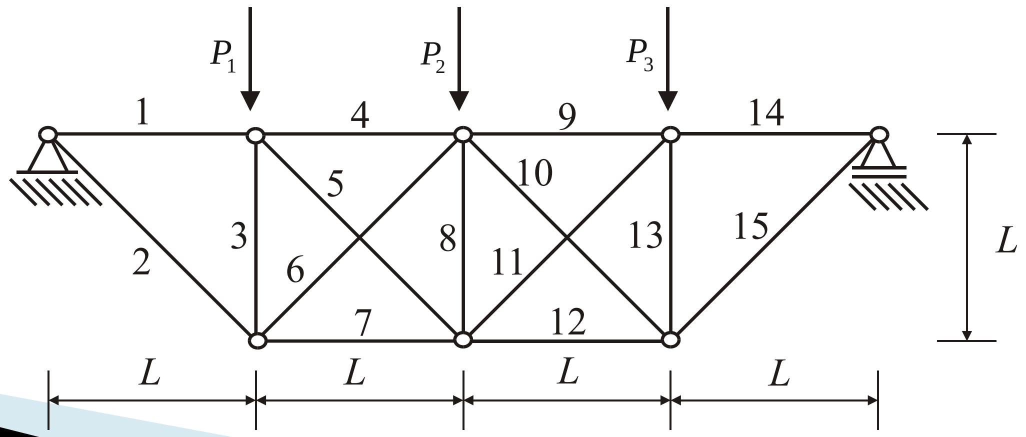


Perturbated forces

$$P = P_0 \pm \Delta P$$

5% uncertainty

No	1	2	3	4	5	6	7	8
ERROR %	10	9,998586	10,00184	10,00126	60,18381	11,67825	9,998955	31,8762
No	9	10	11	12	13	14	15	
ERROR %	10,00126	11,67825	60,18381	9,998955	10,00184	10	9,998586	



Interval equations

$$ax = b \qquad x = \frac{b}{a}$$

Example

$$[1,2]x = [1,4]$$

$$x = ?$$

Interval FEM

- http://en.wikipedia.org/wiki/Interval_finite_element

Differences between intervals and uniformly distributed random variables

Interval

Random variable

Differences between intervals and uniformly distributed random variables

Intervals

$$\mathbf{p}_1 + \ldots + \mathbf{p}_m = \left[\sum_i \underline{p}_i , \sum_i \overline{p}_i \right] = n\left[\underline{p}, \overline{p} \right]$$

$$width(n\mathbf{p}) = n\left(\overline{p} - \underline{p} \right) = n \cdot width(\mathbf{p})$$

Random variables

$$\sigma_{n\mathbf{p}} = \sqrt{\sum_i \sigma_i^2} = \sqrt{n\sigma^2} = \sigma\sqrt{n}$$

$$width(n\mathbf{p}) = \sqrt{n} \cdot width(\mathbf{p})$$

Differences between intervals and uniformly distributed random variables

$$\frac{width(n\mathbf{p}_{\text{int}})}{width(n\mathbf{p}_{rand})} = \frac{n \cdot width(\mathbf{p}_{\text{int}})}{\sqrt{n} \cdot width(\mathbf{p}_{rand})} = \sqrt{n}$$

Example n=100

$$\frac{width(n\mathbf{p}_{\text{int}})}{width(n\mathbf{p}_{rand})} = \frac{n \cdot width(\mathbf{p}_{\text{int}})}{\sqrt{n} \cdot width(\mathbf{p}_{rand})} = \sqrt{100} = 10$$

Partial safety factors for materials

$$S_d = \frac{S_u}{\gamma_m}$$

- S_d – design value
- S_c – characteristic value

Partial safety factor for materials, account for

- (i) the possibility of unfavourable deviation of material strength from the
- characteristic value.
- (ii) the possibility of unfavourable variation of member sizes.
- (iii) the possibility of unfavourable reduction in member strength due to
- fabrication and tolerances.
- (iv) uncertainty in the calculation of strength of the members.

Partial safety factors for loads

$$F_d = \gamma_f F_c$$

- F_d – design value
- F_c – characteristic value

Limit state is a function of safety factors

$$\varphi R > \alpha D + \psi\gamma\left(\alpha_L L + \alpha_Q Q + \alpha_T T\right)$$

$$\varphi R - \left(\alpha D + \psi\gamma\left(\alpha_L L + \alpha_Q Q + \alpha_T T\right)\right) > 0$$

$$L\left(\gamma_i, \theta\right) > 0 \Rightarrow \theta$$

Calibration of safety factors

$$L\left(\gamma_{i\alpha}\right) > 0$$

$$\Updownarrow$$

$$P_f < P_{f0}$$

$\alpha = P_f$ – probability of failure

Probability of failure

- Probability of failure =
- = (number of safe cases)/(number of all cases)

$$P_f = \int_{g(x)<0} f_X(x)\,dx$$

structure fail often

=

structure is not safe

Probabilistic vs nonprobabilistic methods

- Probabilistic methods
  - How often the stucture fail?
- Non-probabilistic methods (worst case design)
  - How big force the structure is able to survive?

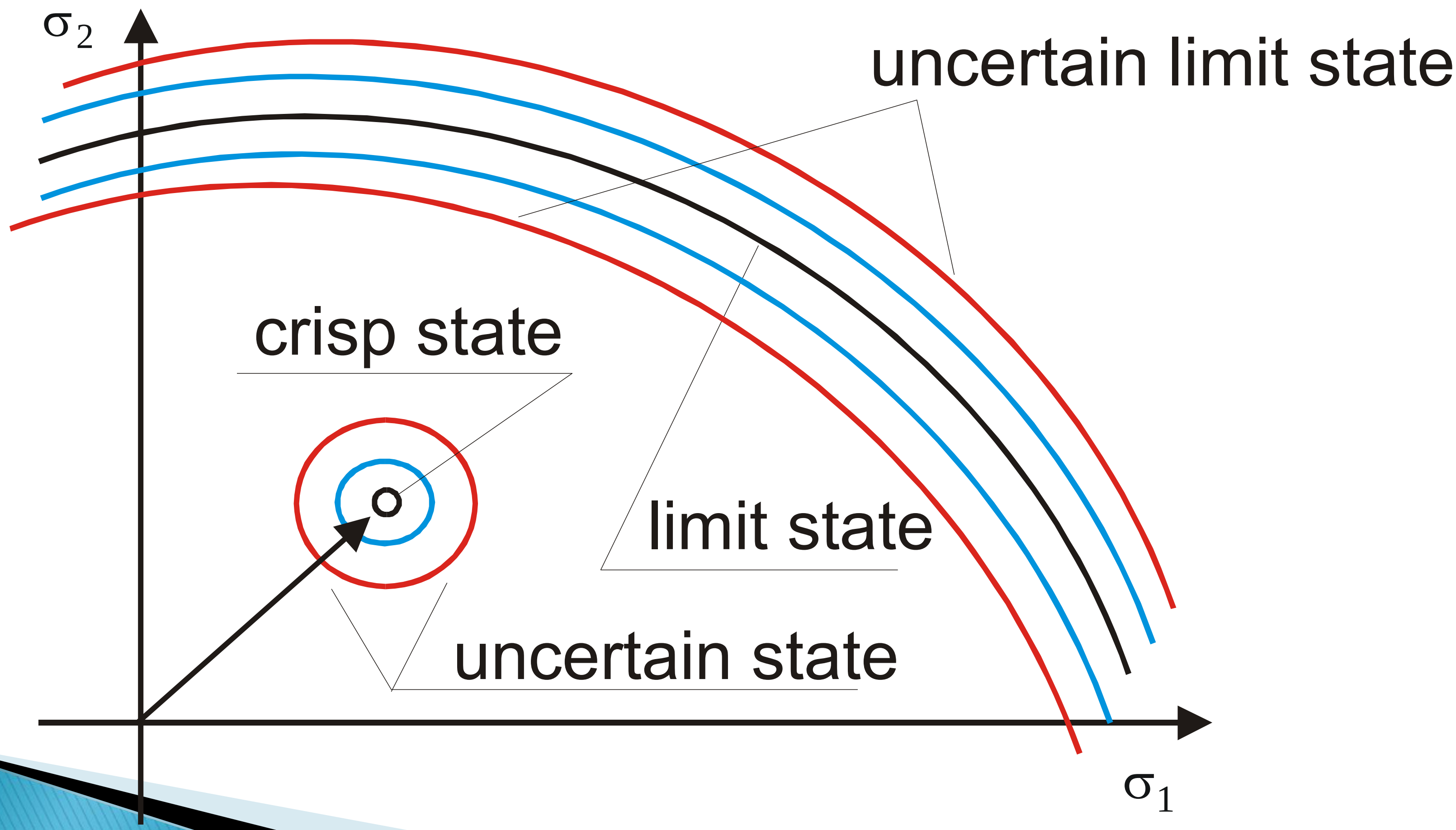
Probabilistic design

$$P_f\left(\theta\right) = P\left\{g(x,\theta) < 0\right\} = \int_{g(x,\theta)<0} f\left(x,\theta\right)dx$$

$$P_f\left(\theta\right) \leq P_{f0} \Rightarrow \theta$$

- Elishakoff I., 2000, Possible limitations of probabilistic methods in engineering. Applied Mechanics Reviews, Vol.53, No.2,pp.19-25
- Does God play dice?

More complicated safety conditions



Uncertain parameters

Set valued random variable

$$h : \Omega \ni \omega \to h(\omega) \subseteq R^n$$

Upper and lower probability

$$Pl(A) = P\{\omega : h(\omega) \cap A \neq \emptyset\}$$

$$Bel(A) = P\{\omega : h(\omega) \subseteq A\}$$

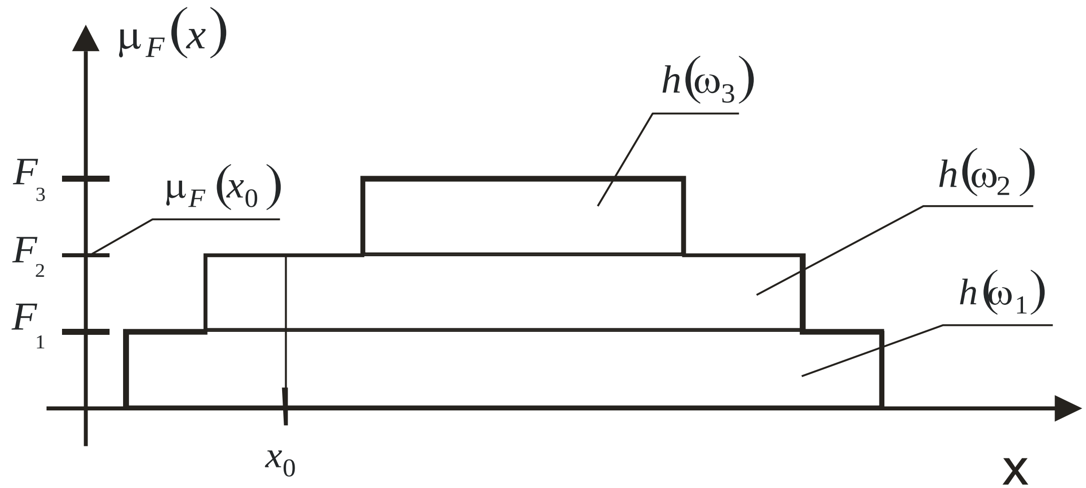
Design under uncertainty

$$Pl(\theta) \leq P_0 \Rightarrow \theta$$

Uncertain parameters

- Nested family of random sets

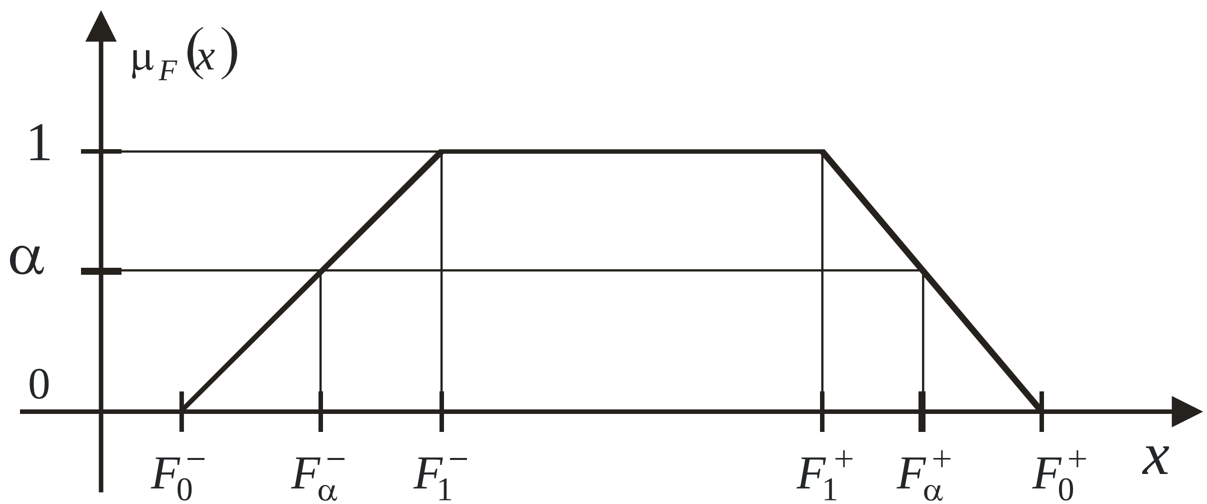
$$h(\omega_1) \supseteq h(\omega_2) \supseteq \ldots \supseteq h(\omega_N)$$



$$\mu_F(x) = P\{\omega : x \in h(\omega)\}$$

Uncertain parameters

- Fuzzy sets



Extension principle

$$\mu_{f(F)}(y) = \sup_{x:y=f(x)} \mu_F(x)$$

Design with fuzzy parameters

$$g(x_F, \theta) \geq 0$$

$$\theta_\alpha = \left\{\theta : g(x, \theta) \geq 0, x \in x_\alpha\right\}$$

$$\mu_\theta\left(\theta\right) = \max\left\{\alpha : \theta \in \theta_\alpha\right\}$$

Fuzzy sets and probability

- http://andrzej.pownuk.com/fuzzy.htm

- Fuzzy approach (the use of grades)
  is similar to the concept of safety factor.

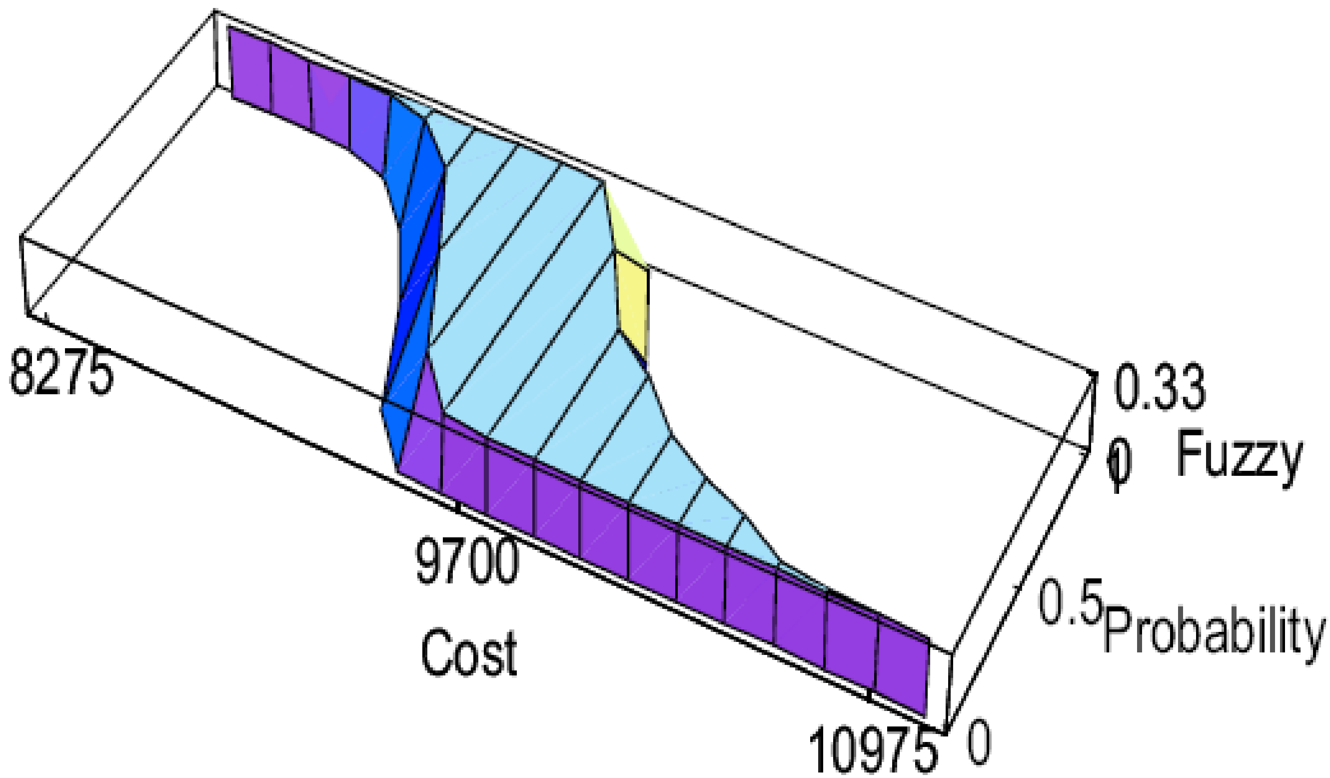
  Because of that fuzzy approach
  is very important.

Alpha cut method

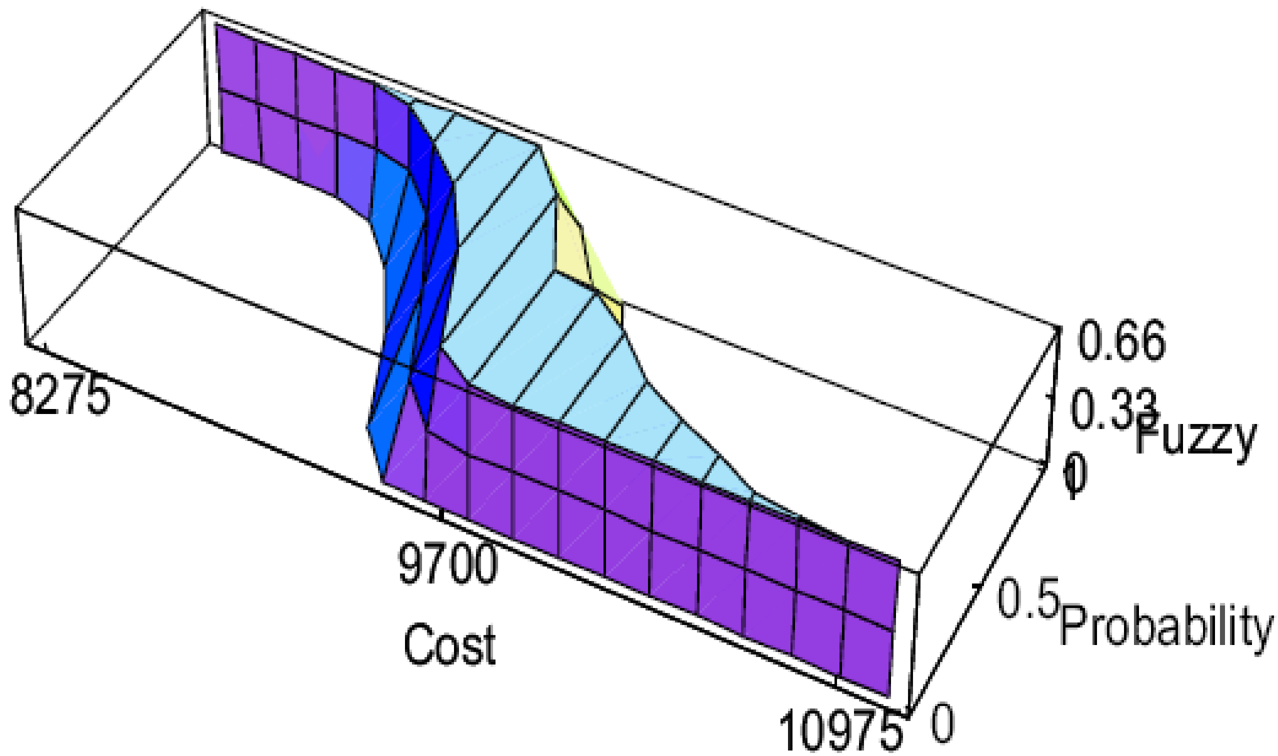
$$\mu_F(x) = \max\left\{\alpha : x \in \left[\underline{F}_\alpha, \bar{F}_\alpha\right]\right\}$$

$$\mu_R(c, p) = \max\left\{\alpha : (p, c) \in R_\alpha\right\}$$

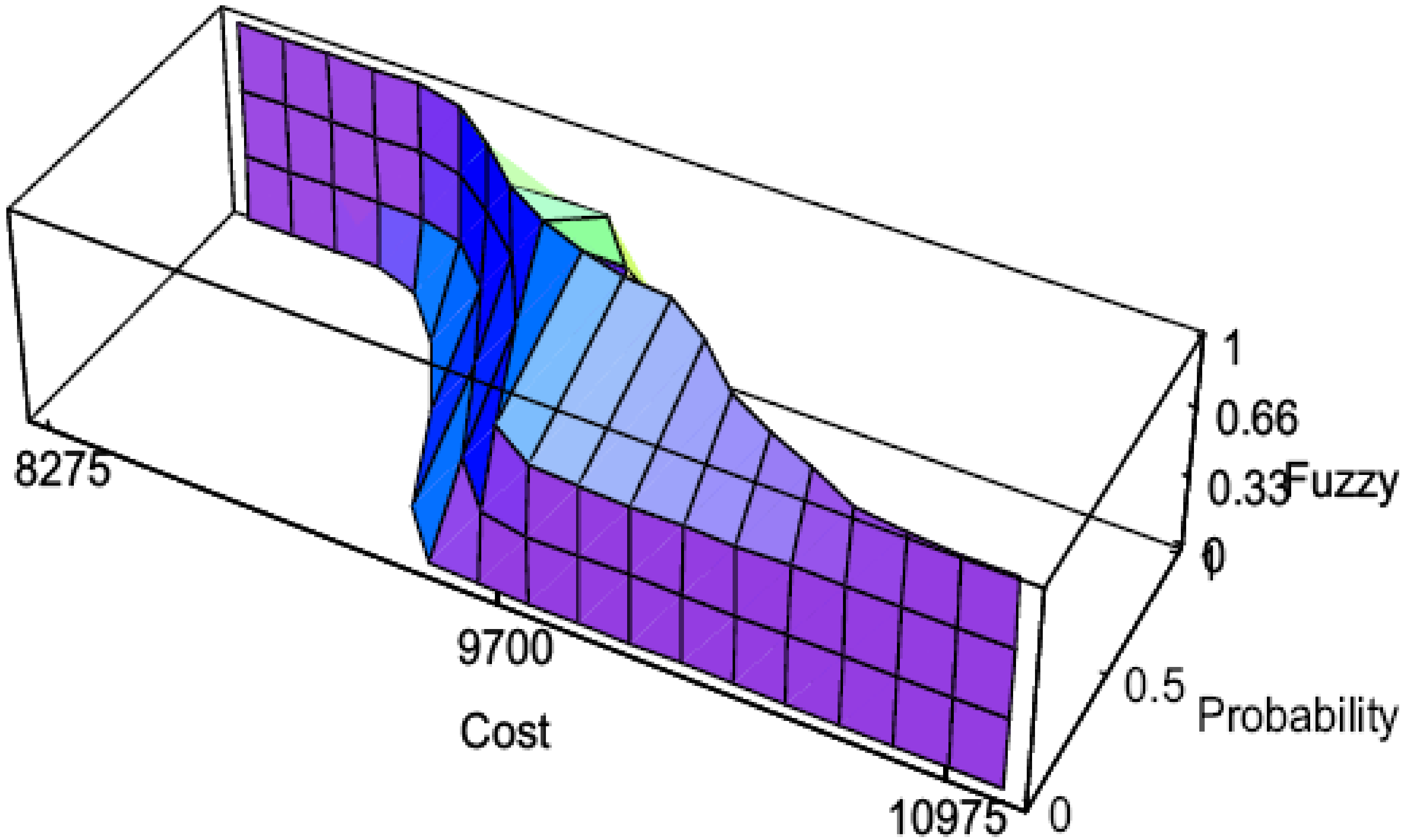
Interval risk curve for different



Interval risk curve for different uncertainty



Interval risk curve for different



Bayesian inference (subjective probability)

$$P(H \mid E) = \frac{P(E \mid H)P(H)}{P(E)}$$

$P(H)$ – is called the *prior probability* of *H* that was inferred before new evidence, *E*, became available

$P(E \mid H)$ – is called the *conditional probability* of seeing the evidence *E* if the hypothesis *H* happens to be true.
It is also called a *likelihood function* when it is considered as a function of *H* for fixed *E*.

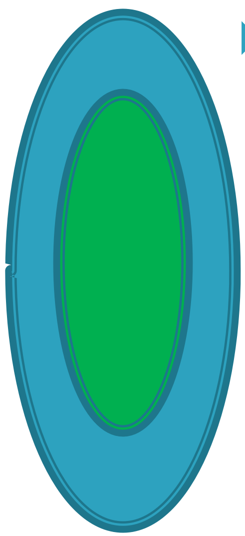
$P(E)$ – marginal probability

$P(H \mid E)$ – is called the *posterior probability* of *H* given *E*.

Cox's theorem

- **Cox's theorem**, named after the physicist Richard Threlkeld Cox, is a derivation of the laws of probability theory from a certain set of postulates. This derivation justifies the so-called "logical" interpretation of probability. As the laws of probability derived by Cox's theorem are applicable to any proposition, logical probability is a type of Bayesian probability. Other forms of Bayesianism, such as the subjective interpretation, are given other justifications.

Rough sets, soft sets
(sets with features)



- First described by Zdzisław I. Pawlak, is a formal approximation of a crisp set (i.e., conventional set) in terms of a pair of sets which give the *lower* and the *upper* approximation of the original set. In the standard version of rough set theory (Pawlak 1991), the lower- and upper-approximation sets are crisp sets, but in other variations, the approximating sets may be fuzzy sets.

Interval and
set valued parameters

Interval parameters

$$p \in \left[\, \underline{p}, \overline{p} \,\right]$$

Design with the interval parameters

$$g(p,\theta) \geq 0$$

$$\boldsymbol{\theta} = \left\{ \theta : g(p,\theta) \geq 0,\ p \in \mathbf{p} \right\}$$

Important papers

- R. E. Moore. Interval Analysis. Prentice-Hall, Englewood Cliffs N. J., 1966
- Neumaier A., 1990, Interval methods for systems of equations, Cambridge University Press, New York
- Ben-Haim Y., Elishakoff I., 1990, Convex Models of Uncertainty in Applied Mechanics. Elsevier Science Publishers, New York
- Buckley J.J., Qy Y., 1990, On using a-cuts to evaluate fuzzy equations. Fuzzy Sets and Systems, Vol.38,pp.309-312

Important papers

- Köylüoglu H.U., A.S. Çakmak, and S. R. K. Nielsen (1995). "Interval Algebra to Deal with Pattern Loading of Structural Uncertainties," ASCE Journal of Engineering Mechanics, 11, 1149–1157

Important papers

- Rump S.M., 1994, Verification methods for dense and sparse systems of equations. J. Herzberger, ed., Topics in Validated Computations. Elsevier Science B.V.,pp.63-135

Important papers

- Muhanna in the paper Muhanna R.L., Mullen R.L., Uncertainty in Mechanics Problems – Interval – Based Approach. Journal of Engineering Mechanics, Vol.127, No.6, 2001, 557–556
- E.Popova, On the Solution of Parametrised Linear Systems. W. Kraemer, J. Wolff von Gudenberg (Eds.): Scientific Computing,Validated Numerics, Interval Methods. Kluwer Acad. Publishers, 2001, pp. 127–138.

Important papers

- I. Skalna, A Method for Outer Interval Solution of Systems of Linear Equations Depending Linearly on Interval Parameters,
Reliable Computing, Volume 12, Number 2, April, 2006, Pages 107–120

Important papers

- Akpan U.O., Koko T.S., Orisamolu I.R., Gallant B.K., Practical fuzzy finite element analysis of structures, Finite Elements in Analysis and Design, 38 (2000) 93–111

- McWilliam, Stewart, 2001
  Anti-optimisation of uncertain structures using interval analysis
  Computers and Structures Volume: 79, Issue: 4, February, 2001, pp. 421–430

Important papers

- Pownuk A., Numerical solutions of fuzzy partial differential equation and its application in computational mechanics, FuzzyPartial Differential Equations and Relational Equations: Reservoir Characterization and Modeling (M. Nikravesh, L. Zadeh and V. Korotkikh, eds.), Studies in Fuzziness and Soft Computing, Physica-Verlag, 2004, pp.308-347

Important papers

- Neumaier A., Clouds, fuzzy sets and probability intervals,
  Reliable Computing 10: 249–272, 2004

- http://andrzej.pownuk.com/IntervalEquations.htm

Examples

- http://webapp.math.utep.edu:8080/~andrzej/php/ansys2interval/
- http://webapp.math.utep.edu/Pages/IntervalFEMExamples.htm
- http://calculus.math.utep.edu/IntervalODE-1.0/default.aspx
- http://calculus.math.utep.edu/AdaptiveTaylorSeries-1.1/default.aspx
- http://andrzej.pownuk.com/silverlight/VibrationsWithIntervalParameters/VibrationsWithIntervalParameters.html

Interval equations

$$ax = b \qquad x = \frac{b}{a}$$

Example

$$[1,2]x = [1,4]$$

$$x = ?$$

Algebraic solution

$$[1,2]x = [1,4]$$

$$x = [1,2]$$

because

$$[1,2][1,2] = [1,4]$$

Algebraic solution

$$[1,4]x = [1,4]$$

$$x = [1,1] = 1$$

because

$$[1,4] \cdot 1 = [1,4]$$

Algebraic solution

$$[1,8]x = [1,4]$$

$$x = ?$$

United solution set

$$[1,2]x = [1,4]$$

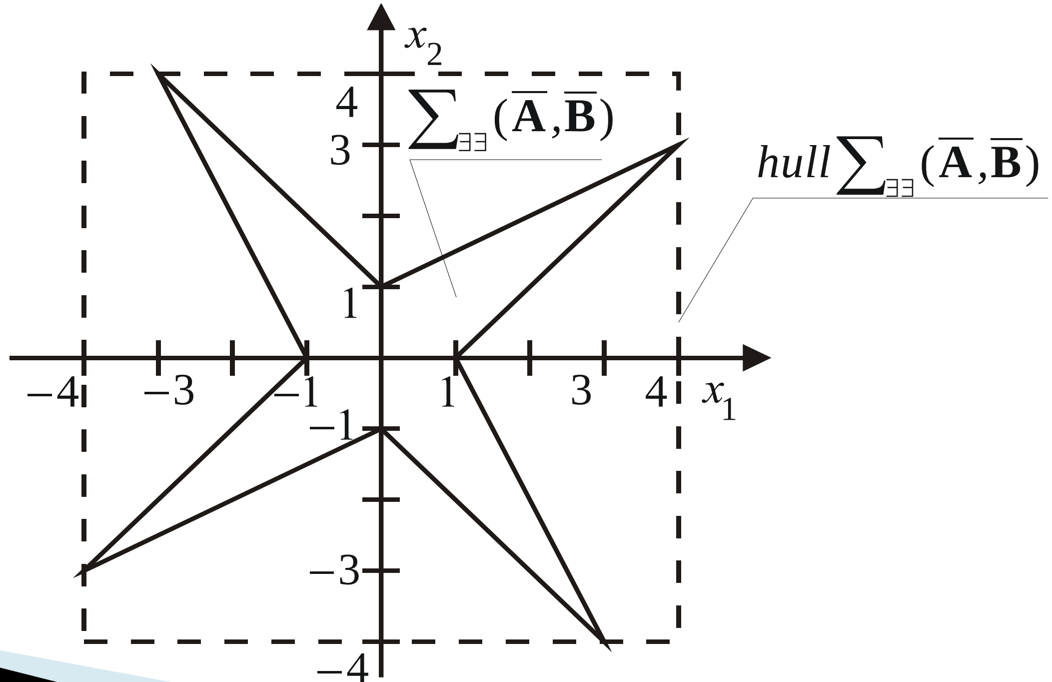
$$\mathbf{x} = \left[\frac{1}{2}, 4\right]$$

because

$$\mathbf{x} = \{x : ax = b, a \in [1,2], b \in [1,4]\}$$

United solution set

$$\begin{bmatrix} [1,2] & [2,4] \\ [2,4] & [1,2] \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} [-1,1] \\ [1,2] \end{bmatrix}$$



Different solution sets

United solution set

$$\sum\nolimits_{\exists\exists}(\mathbf{A},\mathbf{b}) = \{x : \exists A \in \mathbf{A}, \exists b \in \mathbf{b}, Ax = b\}$$

Tolerable solution set

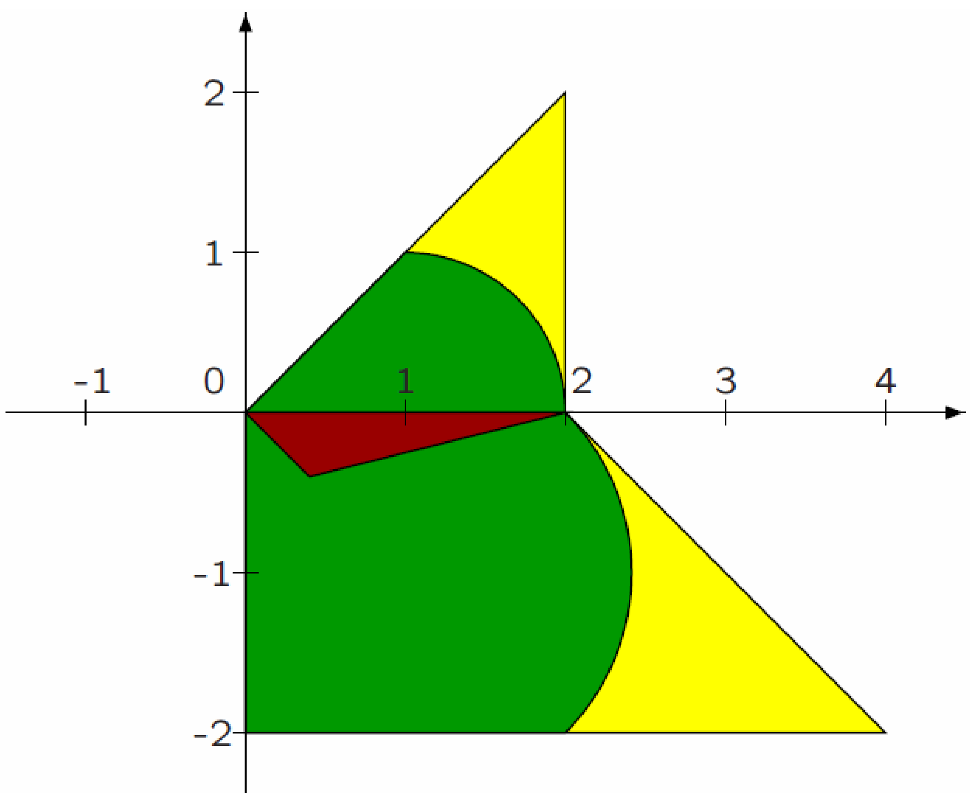
$$\sum\nolimits_{\forall\exists}(\mathbf{A},\mathbf{b}) = \{x : \forall A \in \mathbf{A}, \exists b \in \mathbf{b}, Ax = b\}$$

Controllable solution set

$$\sum\nolimits_{\exists\forall}(\mathbf{A},\mathbf{b}) = \{x : \exists A \in \mathbf{A}, \forall b \in \mathbf{b}, Ax = b\}$$

Other solution sets

$$\boldsymbol{A} = \begin{pmatrix} 1 & [0,1] \\ [0,1] & [-4,-1] \end{pmatrix}, \ \boldsymbol{b} = \begin{pmatrix} [0,2] \\ [0,2] \end{pmatrix},$$



— Ξ_{uni}

— Ξ_{uni}^{sym}

— $\Xi_{tol} = \Xi_{tol}^{sym}$

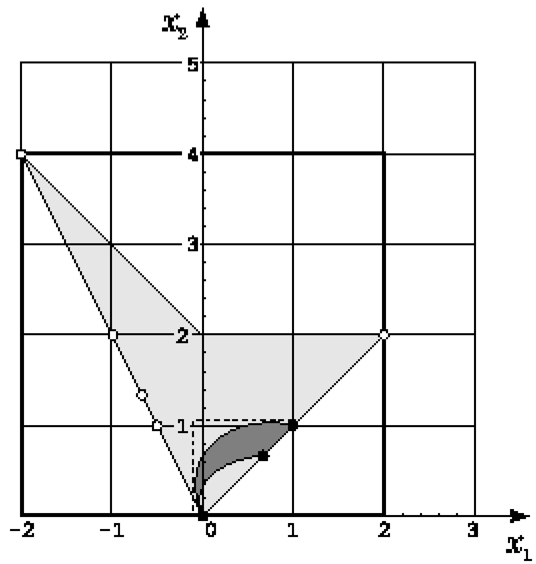
Parametric solution sets

Thus, the *Parametric Solution Set* (PSS) is defined as:

$$\Sigma(\mathbf{A}(\mathbf{p}), \mathbf{b}(\mathbf{p})) = \{\, \tilde{\mathbf{x}} \in \mathbb{R}^n \mid (\exists\, \tilde{\mathbf{p}} \in \mathbf{p})\ \mathbf{A}(\tilde{\mathbf{p}})\tilde{\mathbf{x}} = \mathbf{b}(\tilde{\mathbf{p}}) \,\}.$$

Contrary to the normal case, *Parametric Boundary Solution Set* does not in general contain extremal points of PSS, so at most:

$$\operatorname{hull} \Sigma(\mathbf{A}(\partial\mathbf{p}), \mathbf{b}(\partial\mathbf{p})) \subseteq \operatorname{hull} \Sigma(\mathbf{A}(\mathbf{p}), \mathbf{b}(\mathbf{p})).$$



$\Sigma(\mathbf{A}, \mathbf{b})$ — $\Sigma(\mathbf{A}(\mathbf{p}), \mathbf{b}(\mathbf{p}))$

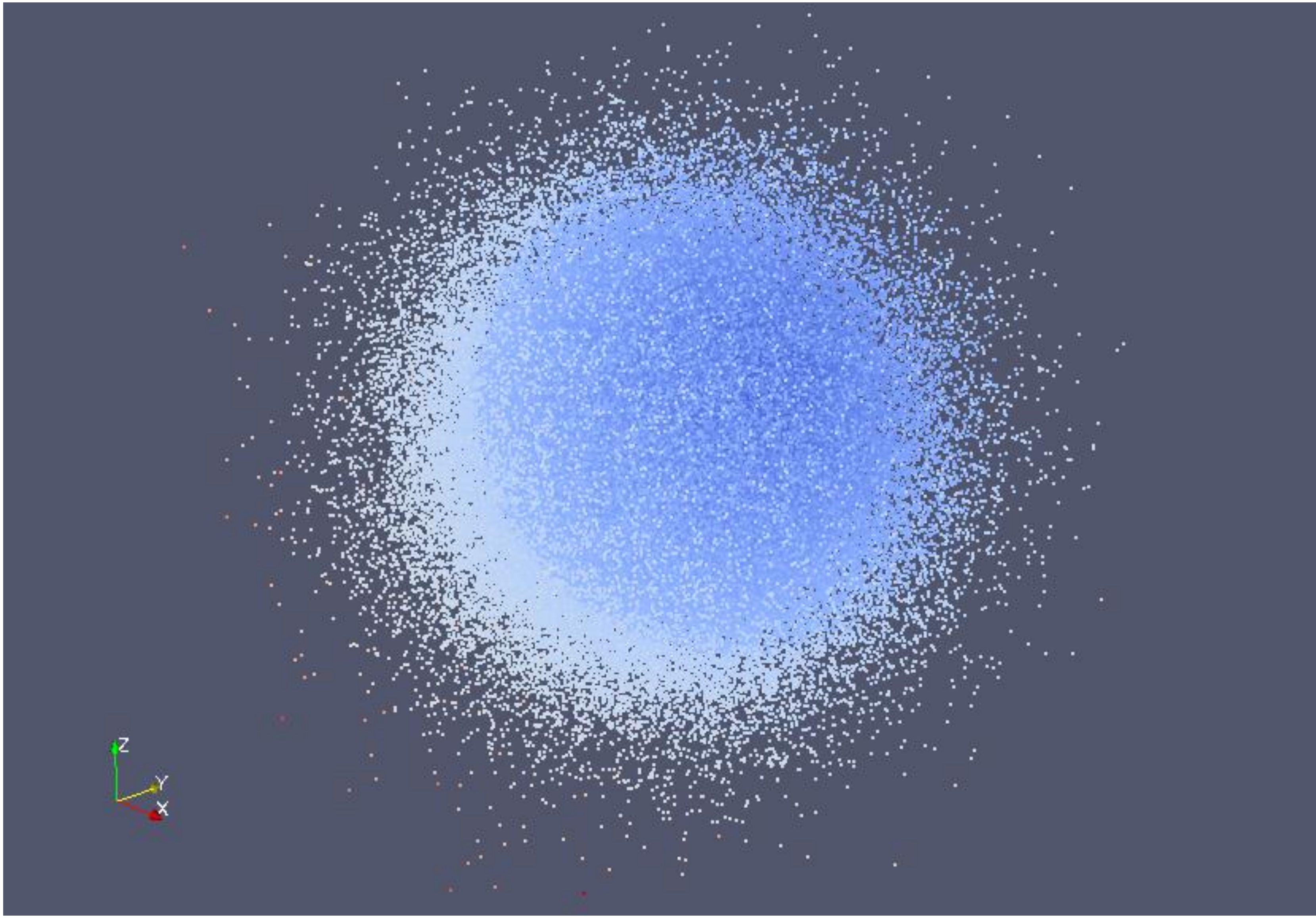
$\Sigma(\partial\mathbf{A}, \partial\mathbf{b})$ — $\Sigma(\mathbf{A}(\partial\mathbf{p}), \mathbf{b}(\partial\mathbf{p}))$

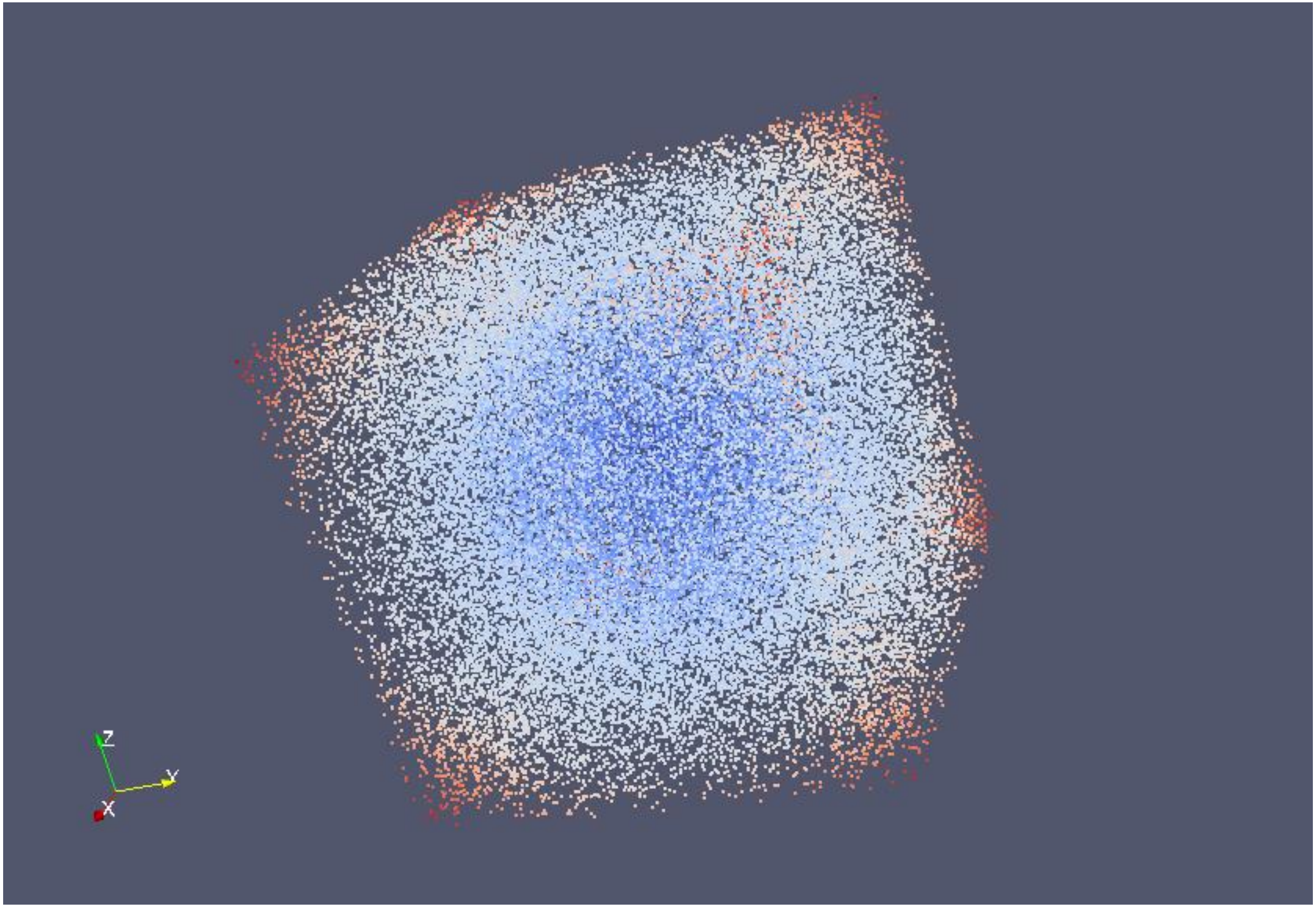
hull $\Sigma(\mathbf{A}, \mathbf{b})$ — hull $\Sigma(\mathbf{A}(\mathbf{p}), \mathbf{b}(\mathbf{p}))$

$$\mathbf{A} = \begin{bmatrix} [0, 1] & [1, 2] \\ -2 & [-1, 2] \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} [0, 2] \\ 0 \end{bmatrix},$$

$$\mathbf{A}(\mathbf{p}) = \begin{bmatrix} p_1 & 1+p_2 \\ -2 & 3p_1-1 \end{bmatrix} \quad \mathbf{b}(\mathbf{p}) = \begin{bmatrix} 2p_1 \\ 0 \end{bmatrix}, \quad p_1, p_2 \in [0, 1]$$

http://www.ippt.gov.pl/~kros/pccmm99/10PSS.html


Z
Y
X


Z
Y
X

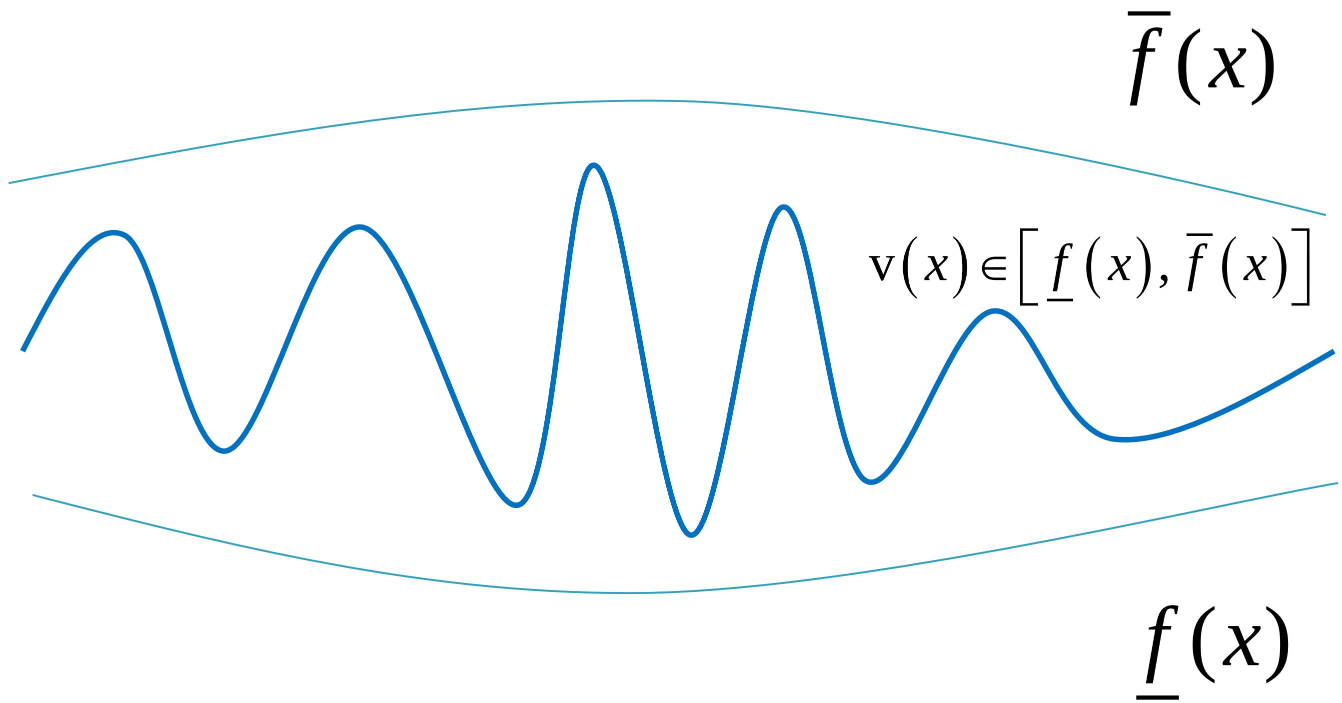
Derivative

$$f(x) = \left[\underline{f}(x), \overline{f}(x) \right]$$

$$f'(x) = \left[\min\left\{ \underline{f}'(x), \overline{f}'(x) \right\}, \max\left\{ \underline{f}'(x), \overline{f}'(x) \right\} \right]$$

What is the definition of the solution of differential equation?

Differentiation of the interval function



$$\mathrm{v}'(x) \notin \left[\min\left\{\underline{f}'(x), \overline{f}'(x)\right\}, \max\left\{\underline{f}'(x), \overline{f}'(x)\right\}\right]$$

What is the definition of the solution
of differential equation?

Dubois D., Prade H., 1987, On Several Definition of the Differentiation of Fuzzy Mapping,
Fuzzy Sets and Systems, Vol.24, pp.117-120

How about integral equations?

Other problems

- Modal interval arithmetic
- Affine arithmetic
- Constrain interval arithmetic
- Ellipsoidal arithmetic
- Convex models (equations with the ellipsoidal parameters)
- General set valued arithmetic
- Fuzzy relational equations
- …. Etc.

Methods of solutions

What does everyone have in mind?

Taylor expansion

$$y\left(p_1,\ldots,p_m\right)\approx y\left(p_{10},\ldots,p_{m0}\right)+\frac{\partial y}{\partial p_1}\left(p_1-p_{10}\right)+\ldots+\frac{\partial y}{\partial p_m}\left(p_m-p_{m0}\right)$$

$$y_0=y\left(p_{10},\ldots,p_{m0}\right)$$

$$\Delta y\approx\left|\frac{\partial y}{\partial p_1}\right|\Delta p_1+\ldots+\left|\frac{\partial y}{\partial p_m}\right|\Delta p_m$$

$$y\in\left[\underline{y},\overline{y}\right]\approx\left[y_0-\Delta y,y_0+\Delta y\right]$$

General optimization methods

- Gradient descent
- Interior point method
- Sequential quadratic programming
- Genetic algorithms
- ...

Interval methods and reliable computing

System of linear interval equations

- Endpoint combination method
- Interval Gauss elimination
- Interval Gauss-Seidel method
- Linear programming method
- Rohn method

- Jiri Rohn, "A Handbook of Results on Interval Linear Problems“,2006 http://www.cs.cas.cz/~rohn/publist/handbook.zip

Oettli and Prager (1964)

$$x \in \sum\nolimits_{\exists\exists}(\mathbf{A},\mathbf{b}) \Leftrightarrow \left| mid(\mathbf{A})x - rad(\mathbf{b}) \right| \le \left(rad(\mathbf{A})x + rad(\mathbf{b}) \right)$$

$$A \in \mathbf{A}, b \in \mathbf{b}$$

- W. Oettli, W. Prager. Compatibility of approximate solution of linear equations with given error bounds for coefficients and right-hand sides. Numer. Math. 6: 405–409, 1964.

Linear programming method

$$\begin{cases} \min x_i \\ \left(mid(\mathbf{A})D_s - rad\left(\mathbf{A}\right)\right) x \le \overline{b} \\ \left(mid(\mathbf{A})D_s + rad\left(\mathbf{A}\right)\right) x \ge \overline{b} \\ x \ge 0 \end{cases} \qquad \begin{cases} \max x_i \\ \left(mid(\mathbf{A})D_s - rad\left(\mathbf{A}\right)\right) x \le \overline{b} \\ \left(mid(\mathbf{A})D_s + rad\left(\mathbf{A}\right)\right) x \ge \overline{b} \\ x \ge 0 \end{cases}$$

- H.U. Koyluoglu, A. Çakmak, S.R.K. Nielsen. Interval mapping in structural mechanics. In: Spanos, ed. Computational Stochastic Mechanics. 125–133. Balkema, Rotterdam 1995.

Rohn's method

$$\begin{cases} A_{rc} = mid(\mathbf{A}) - D_r(rad\mathbf{A})D_c \\ b_r = mid(\mathbf{b}) + D_r rad(\mathbf{b}) \end{cases}$$

$$r = (1, -1, 1, \ldots, 1)^T \in J$$

$$conv \sum\nolimits_{\exists\exists} (\mathbf{A}, \mathbf{b}) = conv \sum\nolimits_{\exists\exists} (\mathbf{A}_{rc}, \mathbf{b}_{rc})$$

$$conv \sum\nolimits_{\exists\exists} (\mathbf{A}_{rc}, \mathbf{b}_{rc}) = conv\{x : A_{rc}x = b_r, r, c \in J\}$$

Rohn's method $2^n \cdot 2^n = 2^{2n}$ Combinatoric solution 2^{n^2+n}

Rohn sign-accord method (RSA)

- For every $r \in J$
- Select $c \in J$ recommended $c = sign\left(mid\left(\mathbf{A} \right)^{-1} b_r \right)$
- Solve $A_{rc} x = b_r$
- If $sign\left(x \right) = c$ then register x and go to next r
- Otherwise find $k = \min\left\{ j : sign\left(x_j \right) \neq c_j \right\}$
- Set $c_k = -c_k$ and go to step 1.

Software

VERSOFT: Verification software in MATLAB / INTLAB

http://uivtx.cs.cas.cz/~rohn/matlab/index.html

-Real data only: Linear systems (rectangular)
-Verified description of all solutions of a system of linear equations
-Verified description of all linear squares solutions of a system of linear equations
-Verified nonnegative solution of a system of linear inequalities
-VERLINPROG for verified nonnegative solution of a system of linear equations
-Real data only: Matrix equations (rectangular)
-See VERMATREQN for verified solution of the matrix equation A*X*B+C*X*D=F (in particular, of the Sylvester or Lyapunov equation)

Etc.

Muhanna and Mullen

Fixed point theory

- Find the solution of $Ax = b$
  - Transform into fixed point equation $g(x) = x$

    $g(x) = x - R(Ax - b) = Rb + (I - RA)x$ (R nonsingular)
  - Brouwer's fixed point theorem

    If $Rb + (I - RA)\mathbf{X} \subseteq \mathrm{int}(\mathbf{X})$

    then $\exists\, x \in \mathbf{X},\ Ax = b$

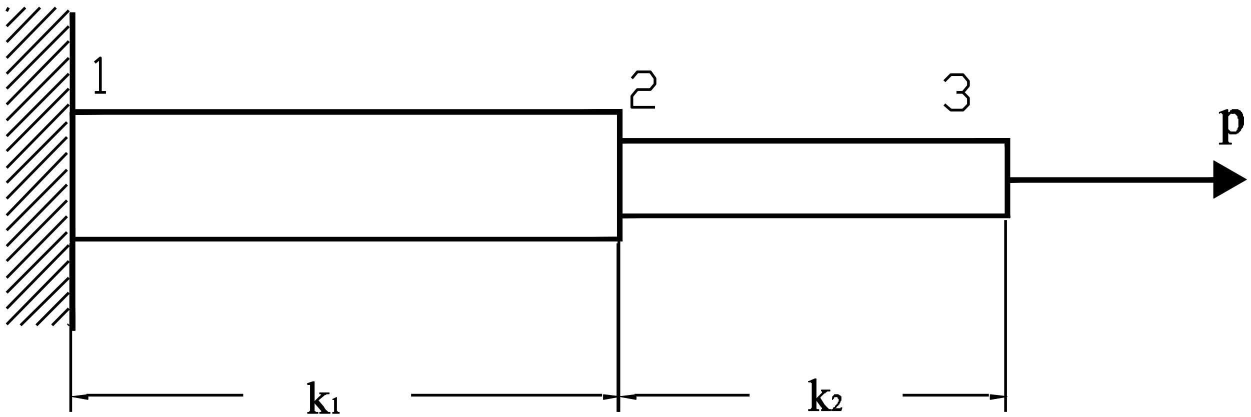
Fixed point theory

- Solve $\mathbf{AX}=\mathbf{b}$
  - Brouwer's fixed point theorem w/ Krawczyk's operator

    If $R\,\mathbf{b} + (I - R\mathbf{A})\,\mathbf{X} \subseteq \mathrm{int}\,(\mathbf{X})$

    then $\Sigma(\mathbf{A}, \mathbf{b}) \subseteq \mathbf{X}$
  - Iteration
    - $\mathbf{X}^{n+1} = R\,\mathbf{b} + (I - R\mathbf{A})\,\epsilon\mathbf{X}^{n}$ (for $n = 0, 1, 2, \ldots$)
    - Stopping criteria: $\mathbf{X}^{n+1} \subseteq \mathrm{int}(\mathbf{X}^{n})$
    - Enclosure: $\Sigma(A, b) \subseteq \mathbf{X}^{n+1}$

Dependency problem



$$\begin{pmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}=\begin{pmatrix} 0 \\ p \end{pmatrix}$$

Dependency problem

$k_1 = [0.9,\ 1.1],\ k_2 = [1.8,\ 2.2],\ p = 1.0$

$$\mathbf{u}_1 = \frac{1}{\mathbf{k}_1} = \frac{1}{[0.9, 1.1]} = [0.91, 1.11]$$

$$\mathbf{u}_2 = \frac{\mathbf{k_1} + \mathbf{k_2}}{\mathbf{k_1 k_2}} = \frac{[0.9, 1.1] + [1.8, 2.2]}{[0.9, 1.1] \times [1.8, 2.2]} = [1.12, 2.04]$$

(overestimation)

$$\mathbf{u}_2' = \frac{1}{\mathbf{k}_1} + \frac{1}{\mathbf{k}_2} = \frac{1}{[0.9, 1.1]} + \frac{1}{[1.8, 2.2]} = [1.36, 1.67]$$

(exact solution)

Dependency problem

$$\mathbf{u}_2 = \frac{\mathbf{k}_1 + \mathbf{k}_2}{\mathbf{k}_1 \mathbf{k}_2}$$

- Two $\mathbf{k}_1$: the same physical quantity
- Interval arithmetic: treat two $\mathbf{k}_1$ as two independent interval quantities having same bounds

Naïve interval finite element

- Replace floating point arithmetic by interval arithmetic
- Over-pessimistic result due to dependency

$$\begin{pmatrix} [2.7,3.3] & [-2.2,-1.8] \\ [-2.2,-1.8] & [1.8,2.2] \end{pmatrix} \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Naïve solution

$$\begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{pmatrix} = \begin{pmatrix} [-110, 112] \\ [-134.5, 137.5] \end{pmatrix}$$

Exact solution

$$\begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{pmatrix} = \begin{pmatrix} [0.91, 1.11] \\ [1.36, 1.67] \end{pmatrix}$$

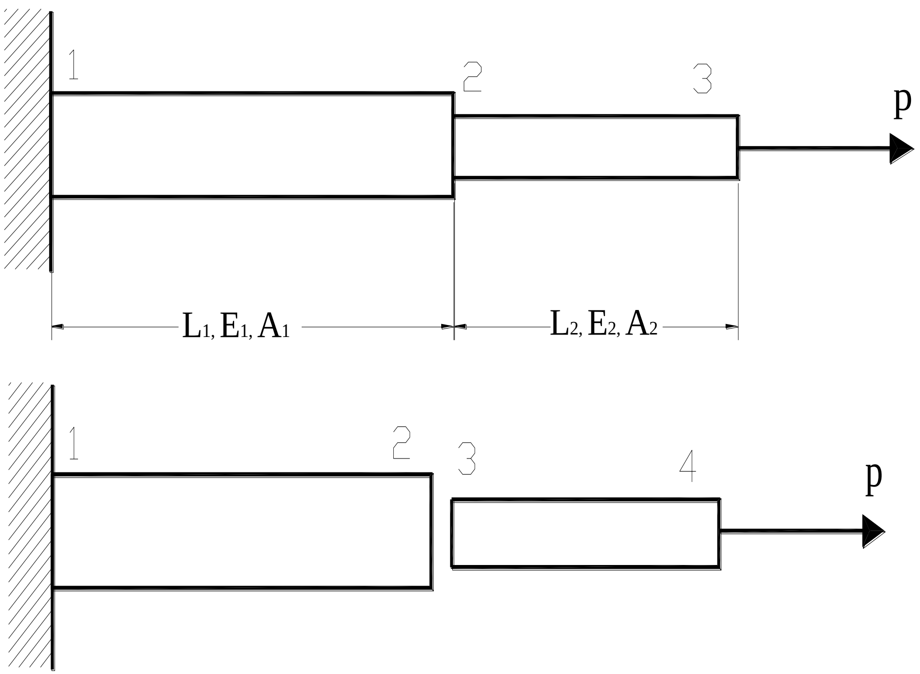
Dependency problem

- How to reduce overestimation?
  - Manipulate the expression to reduce multiple occurrence
  - Trace the sources of dependency

$$\begin{pmatrix} \mathbf{k}_1 + \mathbf{k}_2 & -\mathbf{k}_2 \\ -\mathbf{k}_2 & \mathbf{k}_2 \end{pmatrix} \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{p} \end{pmatrix}$$

Present formulation

- Element-by-Element
  - *K*: diagonal matrix, singular



Present formulation

- Element-by-element method
  - Element stiffness: $\mathbf{K}_i = \breve{K}_i(I + \mathbf{d}_i)$

$$\mathbf{K}_1 = \begin{pmatrix} \frac{\mathbf{E_1}A_1}{L_1} & -\frac{\mathbf{E_1}A_1}{L_1} \\ -\frac{\mathbf{E_1}A_1}{L_1} & \frac{\mathbf{E_1}A_1}{L_1} \end{pmatrix} = \begin{pmatrix} \frac{\breve{E}_1A_1}{L_1} & -\frac{\breve{E}_1A_1}{L_1} \\ -\frac{\breve{E}_1A_1}{L_1} & \frac{\breve{E}_1A_1}{L_1} \end{pmatrix} \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \boldsymbol{\alpha}_1 & 0 \\ 0 & \boldsymbol{\alpha}_1 \end{pmatrix} \right)$$

  - System stiffness: $\mathbf{K} = \breve{K}(I + \mathbf{d})$

Present formulation

- Lagrange Multiplier method
  - With the constraints: $CU - t = 0$
  - Lagrange multipliers: λ

$$\begin{pmatrix} K & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} u \\ \lambda \end{pmatrix} = \begin{pmatrix} p \\ t \end{pmatrix} \Rightarrow \begin{pmatrix} \mathbf{K} & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \mathbf{p} \\ 0 \end{pmatrix}$$

Present formulation

- System equation: $\mathbf{Ax} = \mathbf{b}$

$$\begin{pmatrix} \mathbf{K} & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \mathbf{p} \\ 0 \end{pmatrix}$$

rewrite as:

$$(\breve{A} + S\mathbf{D})\mathbf{x} = \mathbf{b}$$

$$\left[\begin{pmatrix} \breve{k} & C^T \\ C & 0 \end{pmatrix} + \begin{pmatrix} \breve{k} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{d} & 0 \\ 0 & 0 \end{pmatrix} \right] \begin{pmatrix} \mathbf{u} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \mathbf{p} \\ 0 \end{pmatrix}$$

Present formulation

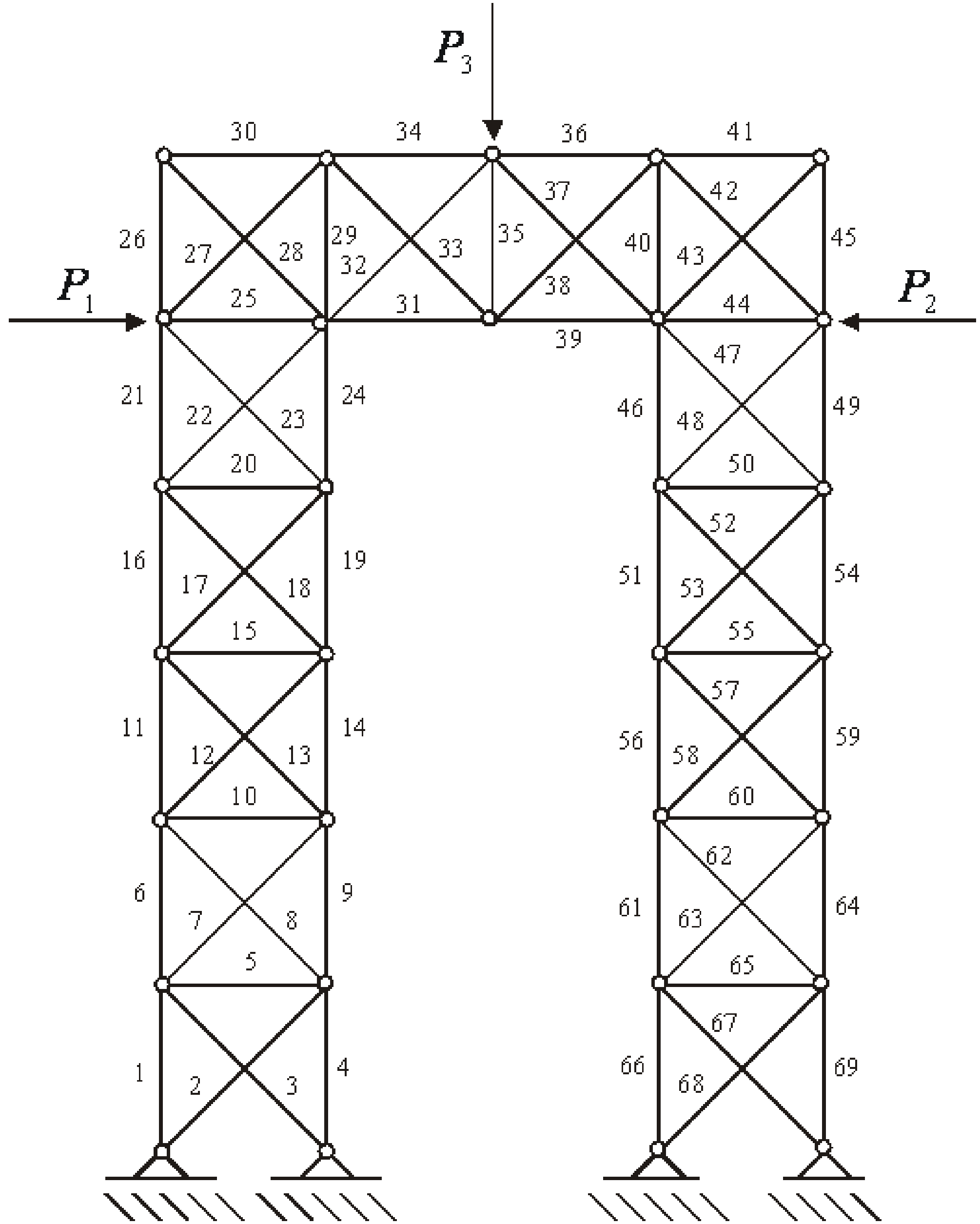
- $\mathbf{x} = [\mathbf{u}, \boldsymbol{\lambda}]^T$, $\mathbf{u}$ is the displacement vector
- Calculate element forces
  - Conventional FEM: $F = k\,u$ (overestimation)
  - Present formulation: $\mathbf{K}\mathbf{u} = \mathbf{P} - C^T \boldsymbol{\lambda}$

    $\boldsymbol{\lambda} = L\mathbf{x}$, $\mathbf{p} = N\mathbf{b}$

    $\mathbf{P} - C^T \boldsymbol{\lambda} = \mathbf{p} - C^T L(\mathbf{x}^{*n+1} + x_0)$

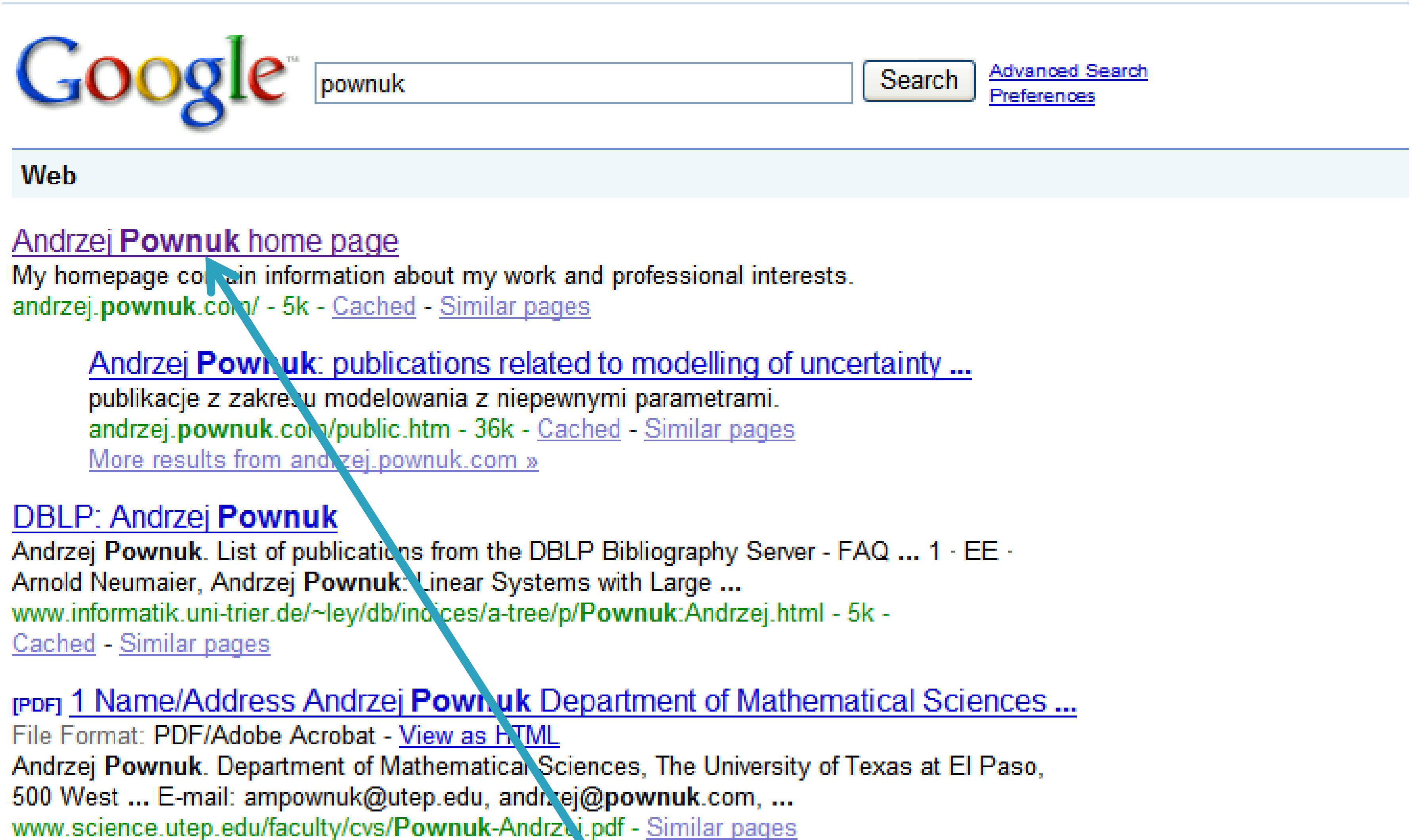
    $\mathbf{P} - C^T \boldsymbol{\lambda} = N\mathbf{b} - C^T L(R\mathbf{b} - RS\,\mathbf{M}^n \boldsymbol{\delta})$

    $\mathbf{P} - C^T \boldsymbol{\lambda} = (N - C^T LR)\mathbf{b} + C^T LRS\,\mathbf{M}^n \boldsymbol{\delta}$



P3
P1
P2
30
34
36
41
26
27
28
29
32
33
35
37
38
40
42
43
45
25
31
39
44
21
22
23
24
20
46
47
48
49
50
16
17
18
19
15
51
52
53
54
55
11
12
13
14
10
56
57
58
59
60
6
7
8
9
5
61
62
63
64
65
1
2
3
4
66
67
68
69

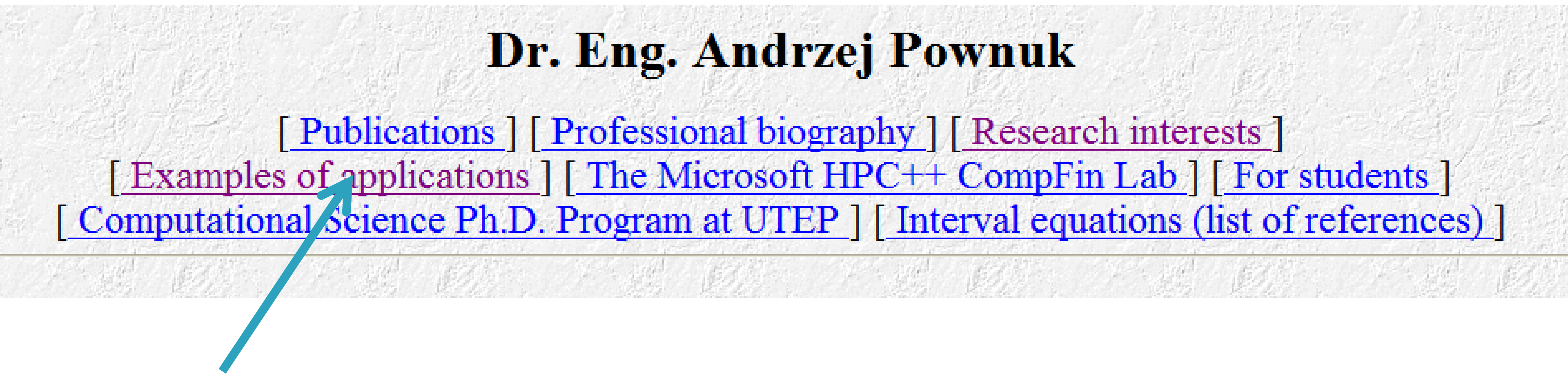
No	Error %	No	Error %	No	Error %	No	Error %
1	107.057 %	21	42.4163 %	41	109.399 %	61	31.4828 %
2	78.991 %	22	34.0332 %	42	20.0367 %	62	95.903 %
3	38.2972 %	23	9.30111 %	43	109.399 %	63	48.1069 %
4	52.0345 %	24	100.427 %	44	0.833207 %	64	68.1526 %
5	22.7834 %	25	0.833207 %	45	116.216 %	65	22.7834 %
6	68.1526 %	26	116.216 %	46	100.427 %	66	52.0345 %
7	95.903 %	27	20.0367 %	47	34.0332 %	67	78.991 %
8	48.1069 %	28	116.216 %	48	9.30111 %	68	38.2972 %
9	31.4828 %	29	48.1793 %	49	42.4163 %	69	107.057 %
10	22.6152 %	30	116.216 %	50	15.3219 %		
11	38.0037 %	31	8.87714 %	51	33.6609 %		
12	91.4489 %	32	19.1787 %	52	119.633 %		
13	45.9704 %	33	35.7319 %	53	47.414 %		
14	11.913 %	34	18.7494 %	54	9.99375 %		
15	24.3824 %	35	6.38613 %	55	24.3824 %		
16	9.99375 %	36	18.7494 %	56	11.913 %		
17	119.633 %	37	19.1787 %	57	91.4489 %		
18	56.7357 %	38	35.7319 %	58	45.9704 %		
19	33.6609 %	39	8.87714 %	59	38.0037 %		
20	15.3219 %	40	48.1793 %	60	22.6152 %		


Google
pownuk
Search
Advanced Search
Preferences
Web
Andrzej Pownuk home page
My homepage co[ntain] information about my work and professional interests.
andrzej.pownuk.com/ - 5k - Cached - Similar pages
Andrzej Pownuk: publications related to modelling of uncertainty ...
publikacje z zakresu modelowania z niepewnymi parametrami.
andrzej.pownuk.com/public.htm - 36k - Cached - Similar pages
More results from andrzej.pownuk.com »
DBLP: Andrzej Pownuk
Andrzej Pownuk. List of publications from the DBLP Bibliography Server - FAQ ... 1 · EE ·
Arnold Neumaier, Andrzej Pownuk: Linear Systems with Large ...
www.informatik.uni-trier.de/~ley/db/indices/a-tree/p/Pownuk:Andrzej.html - 5k -
Cached - Similar pages
[PDF] 1 Name/Address Andrzej Pownuk Department of Mathematical Sciences ...
File Format: PDF/Adobe Acrobat - View as HTML
Andrzej Pownuk. Department of Mathematical Sciences, The University of Texas at El Paso,
500 West ... E-mail: ampownuk@utep.edu, andrzej@pownuk.com, ...
www.science.utep.edu/faculty/cvs/Pownuk-Andrzej.pdf - Similar pages

Click here

Web application

http://andrzej.pownuk.com/



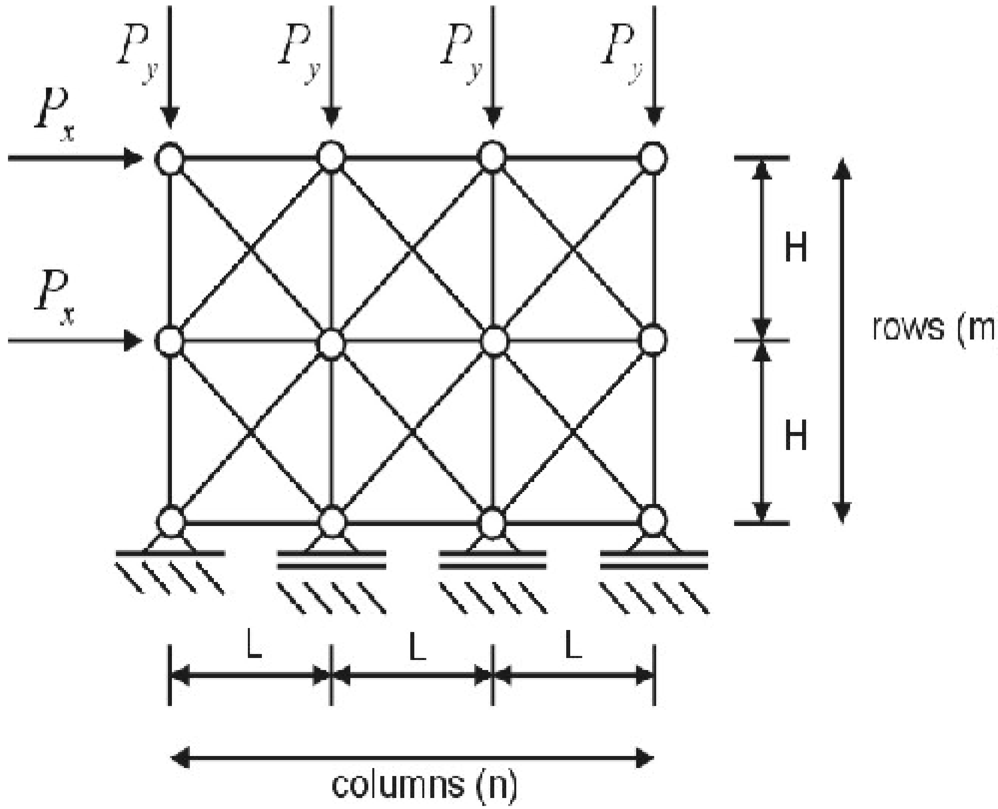
Click here

Truss structure with interval parameters

Describe the structure

2	Number of rows (n)
3	Number of columns (m)
2	L [m]
1	H [m]
10	Px [kN]
10	Py [kN]
210E9	E [N/m^2]
0.0025	A [m^2]
5	dE [%]
5	dA [%]

Generete the code



Monotonicity of the solution

- Monotone solution

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} p_1 + p_2 \\ p_2 \end{bmatrix}$$

$$u_1 = \frac{p_1}{2} + p_2,\ u_2 = \frac{p_1}{2}$$

$$u^2 - p^4 = 0$$

- Non-monotone solution

$$u_1 = p^2, u_2 = -p^2$$

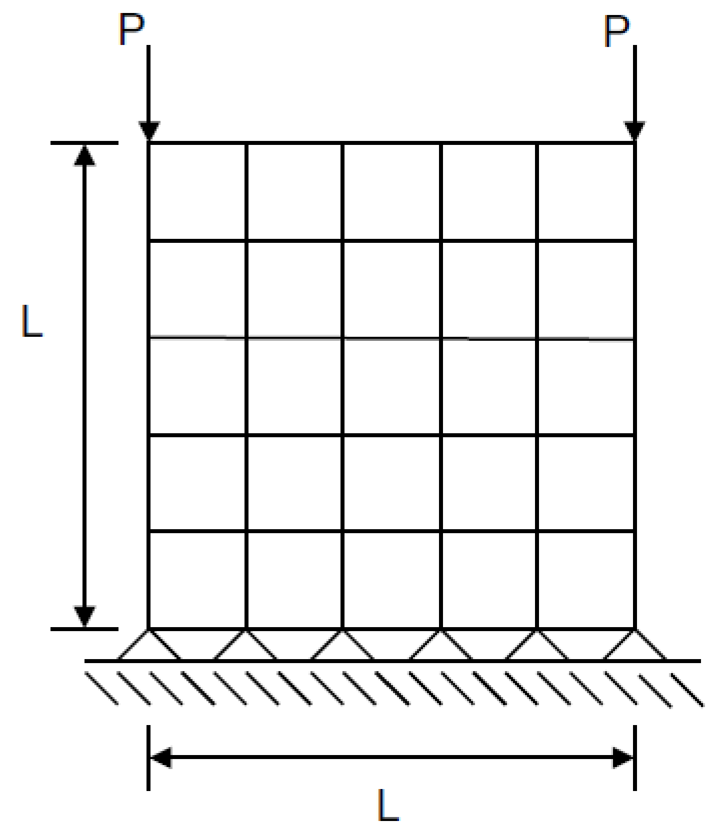
Interval solution for monotone functions (gradient descent method)

If $\dfrac{\partial u}{\partial p} \geq 0$ then $p^{min} = \underline{p}, p^{max} = \overline{p}$

If $\dfrac{\partial u}{\partial p} < 0$ then $p^{min} = \overline{p}, p^{max} = \underline{p}$

$$\underline{u} = u(p^{min}), \overline{u} = u(p^{max})$$

Monotonicity of the solution



Plane stress

$$\begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix} = \begin{bmatrix} \partial/\partial x & 0 \\ 0 & \partial/\partial y \\ \partial/\partial y & \partial/\partial x \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix},$$

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix},$$

$$\begin{bmatrix} \partial/\partial x & 0 & \partial/\partial y \\ 0 & \partial/\partial y & \partial/\partial x \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} + \begin{bmatrix} b_x \\ b_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Interval solution
Verification of the results by using search method with 3 intermediate points

Table VIII. Displacements in the case of 4 (2x2) finite elements and 5% uncertainty

	Combinatoric	Combinatoric	Gradient free	Gradient free		
DOF	$\underline{u}$	$\overline{u}$	$\underline{u}$	$\overline{u}$	Error $\underline{u}$ %	Error $\overline{u}$ %
5	-8.770130E-23	1.113670E-22	-8.770130E-23	1.113670E-22	0.000000E+00	0.000000E+00
6	-4.404760E-07	-3.605710E-07	-4.404760E-07	-3.605710E-07	0.000000E+00	0.000000E+00
7	4.822140E-08	5.890760E-08	4.822140E-08	5.890760E-08	0.000000E+00	0.000000E+00
8	-6.087950E-07	-4.983560E-07	-6.087950E-07	-4.983560E-07	0.000000E+00	0.000000E+00
11	-5.890760E-08	-4.822140E-08	-5.890760E-08	-4.822140E-08	0.000000E+00	0.000000E+00
12	-6.087950E-07	-4.983560E-07	-6.087950E-07	-4.983560E-07	0.000000E+00	0.000000E+00
13	-1.851990E-23	2.469330E-22	-1.851990E-23	2.469330E-22	0.000000E+00	0.000000E+00
14	-3.540790E-07	-2.898470E-07	-3.540790E-07	-2.898470E-07	0.000000E+00	0.000000E+00
15	-4.815970E-07	-3.942320E-07	-4.815970E-07	-3.942320E-07	0.000000E+00	0.000000E+00
16	-1.840780E-06	-1.506850E-06	-1.840780E-06	-1.506850E-06	0.000000E+00	0.000000E+00
17	3.942320E-07	4.815970E-07	3.942320E-07	4.815970E-07	0.000000E+00	0.000000E+00
18	-1.840780E-06	-1.506850E-06	-1.840780E-06	-1.506850E-06	0.000000E+00	0.000000E+00

Interval solution

Table X. Displacements in the case of 9 (3x3) finite elements and 5% uncertainty

	Combinatoric		Gradient free			
DOF	$\underline{u}$	$\overline{u}$	$\underline{u}$	$\overline{u}$	Error $\underline{u}$ %	Error $\overline{u}$ %
5	2.991120E-09	4.758950E-08	2.991120E-09	4.758950E-08	0.000000E+00	0.000000E+00
6	-3.401560E-07	-2.737550E-07	-3.401560E-07	-2.738240E-07	0.000000E+00	2.520502E-02
7	-2.700300E-08	3.178720E-08	-2.681040E-08	3.178720E-08	7.132541E-01	0.000000E+00
8	-4.259370E-07	-2.781400E-07	-4.259370E-07	-2.781400E-07	0.000000E+00	0.000000E+00
11	-4.758950E-08	-2.991120E-09	-4.758950E-08	-2.991120E-09	0.000000E+00	0.000000E+00
12	-3.401560E-07	-2.737550E-07	-3.401560E-07	-2.738240E-07	0.000000E+00	2.520502E-02
15	-3.178720E-08	2.700300E-08	-3.178720E-08	2.681040E-08	0.000000E+00	7.132541E-01
16	-4.259370E-07	-2.781400E-07	-4.259370E-07	-2.781400E-07	0.000000E+00	0.000000E+00
17	-8.584880E-09	1.665680E-07	-8.584880E-09	1.665680E-07	0.000000E+00	0.000000E+00
18	-5.498820E-07	-4.324890E-07	-5.498820E-07	-4.324890E-07	0.000000E+00	0.000000E+00

Interval solution

Table XI. Combinations of parameters which correspond to the interval displacements 9 elements (3x3) and 5% uncertainty

	Combinatoric	Combinatoric	Gradient free	Gradient free		
DOF	$\underline{u}$	$\overline{u}$	$\underline{u}$	$\overline{u}$	$\underline{u}$	$\overline{u}$
5	0,1,1,0,1,0,1,1,1,0,1	1,0,0,1,0,1,0,0,0,1,0	0,1,1,0,1,0,1,1,1,0,1	1,0,0,1,0,1,0,0,0,1,0	0	0
6	0,0,0,0,1,0,1,1,1,0,1	1,1,1,$\boxed{0}$,0,1,0,0,0,1,0	0,0,0,0,1,0,1,1,1,0,1	1,1,1,$\boxed{1}$,0,1,0,0,0,1,0	0	1
7	0,0,1,0,1,0,1,1,1,0,1	1,$\boxed{0}$,0,1,0,1,0,0,0,1,0	0,$\boxed{1}$,1,0,1,0,1,1,1,0,1	1,0,0,1,0,1,0,0,0,1,0	1	0
8	0,0,1,1,0,1,0,0,0,0,1	1,1,0,0,1,0,1,1,1,1,0	0,0,1,1,0,1,0,0,0,0,1	1,1,0,0,1,0,1,1,1,1,0	0	0
11	0,0,1,1,0,1,0,0,0,0,1	1,1,0,0,1,0,1,1,1,1,0	0,0,1,1,0,1,0,0,0,0,1	1,1,0,0,1,0,1,1,1,1,0	0	0
12	0,0,0,0,1,0,1,1,1,1,0	1,1,1,1,0,$\boxed{0}$,0,0,0,0,1	0,0,0,0,1,0,1,1,1,1,0	1,1,1,1,0,$\boxed{1}$,0,0,0,0,1	0	1
15	0,0,1,1,0,1,0,0,0,0,1	1,$\boxed{0}$,0,0,1,0,1,1,1,1,0	0,0,1,1,0,1,0,0,0,0,1	1,$\boxed{1}$,0,0,1,0,1,1,1,1,0	0	1
16	1,0,0,1,0,1,0,0,0,1,0	0,1,1,0,1,0,1,1,1,0,1	1,0,0,1,0,1,0,0,0,1,0	0,1,1,0,1,0,1,1,1,0,1	0	0
17	0,1,1,1,1,1,0,1,0,0,1	1,0,0,0,0,0,1,0,1,1,0	0,1,1,1,1,1,0,1,0,0,1	1,0,0,0,0,0,1,0,1,1,0	0	0
18	0,0,0,0,0,0,1,1,1,0,1	1,1,1,1,1,1,0,0,0,1,0	0,0,0,0,0,0,1,1,1,0,1	1,1,1,1,1,1,0,0,0,1,0	0	0

Interval solution

Table XII. Interval displacements for 10% uncertainty and 9 finite elements

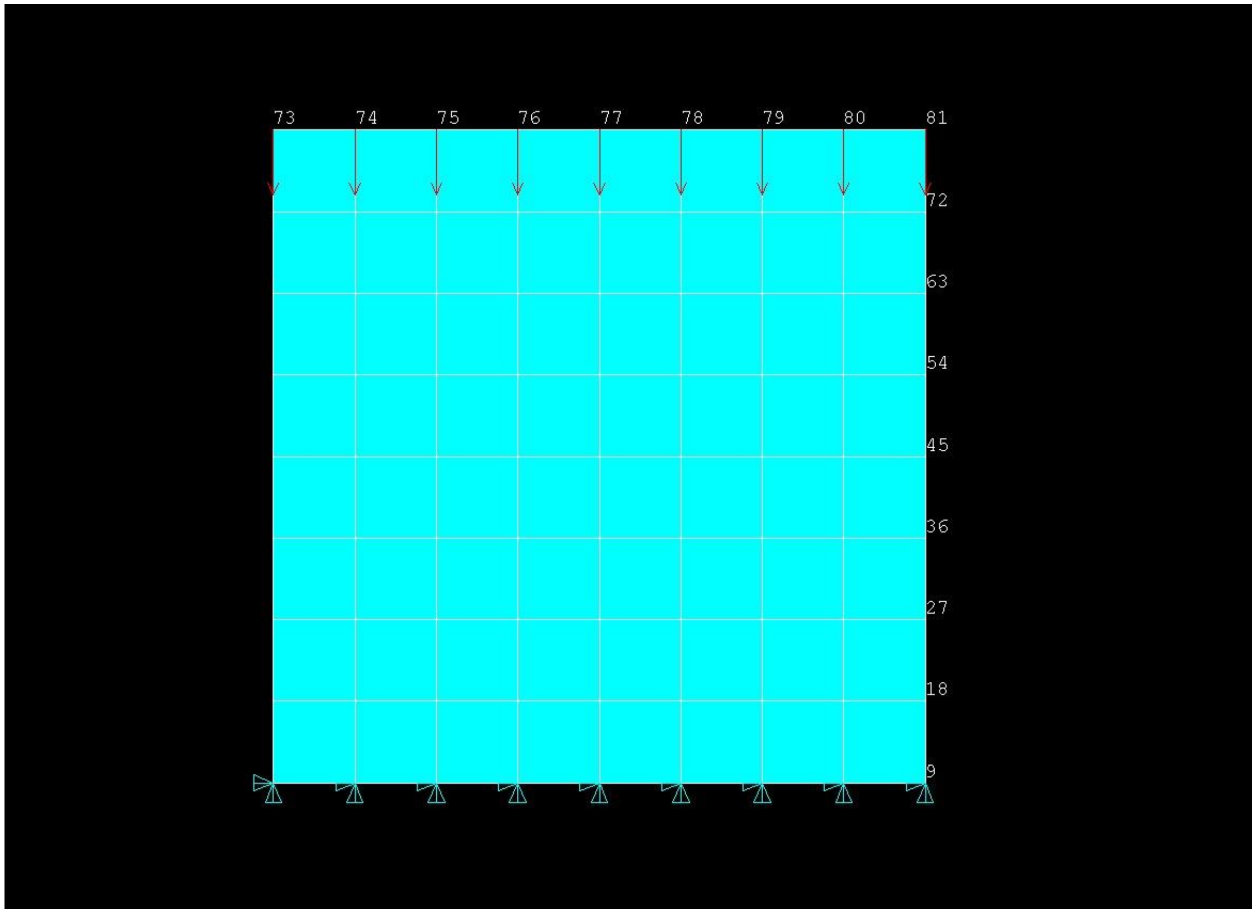
DOF	$\underline{u}$	$\overline{u}$	$\underline{u}$	$\overline{u}$	Error $\underline{u}$ %	Error $\overline{u}$ %
5	-1.934900E-08	7.022680E-08	-1.934900E-08	7.022470E-08	0.000000E+00	2.990311E-03
6	-3.796920E-07	-2.451030E-07	-3.796920E-07	-2.458330E-07	0.000000E+00	2.978340E-01
7	-5.781790E-08	6.089000E-08	-5.677190E-08	6.089000E-08	1.809128E+00	0.000000E+00
8	-5.089660E-07	-2.112750E-07	-5.089660E-07	-2.112750E-07	0.000000E+00	0.000000E+00
11	-7.022680E-08	1.934900E-08	-7.022470E-08	1.934900E-08	2.990311E-03	0.000000E+00
12	-3.796920E-07	-2.451030E-07	-3.796920E-07	-2.458330E-07	0.000000E+00	2.978340E-01
15	-6.089000E-08	5.781790E-08	-6.089000E-08	5.677190E-08	0.000000E+00	1.809128E+00
16	-5.089660E-07	-2.112750E-07	-5.089660E-07	-2.112750E-07	0.000000E+00	0.000000E+00
17	-9.469690E-08	2.590120E-07	-9.315200E-08	2.590120E-07	1.631416E+00	0.000000E+00
18	-6.187590E-07	-3.820950E-07	-6.187590E-07	-3.820950E-07	0.000000E+00	0.000000E+00

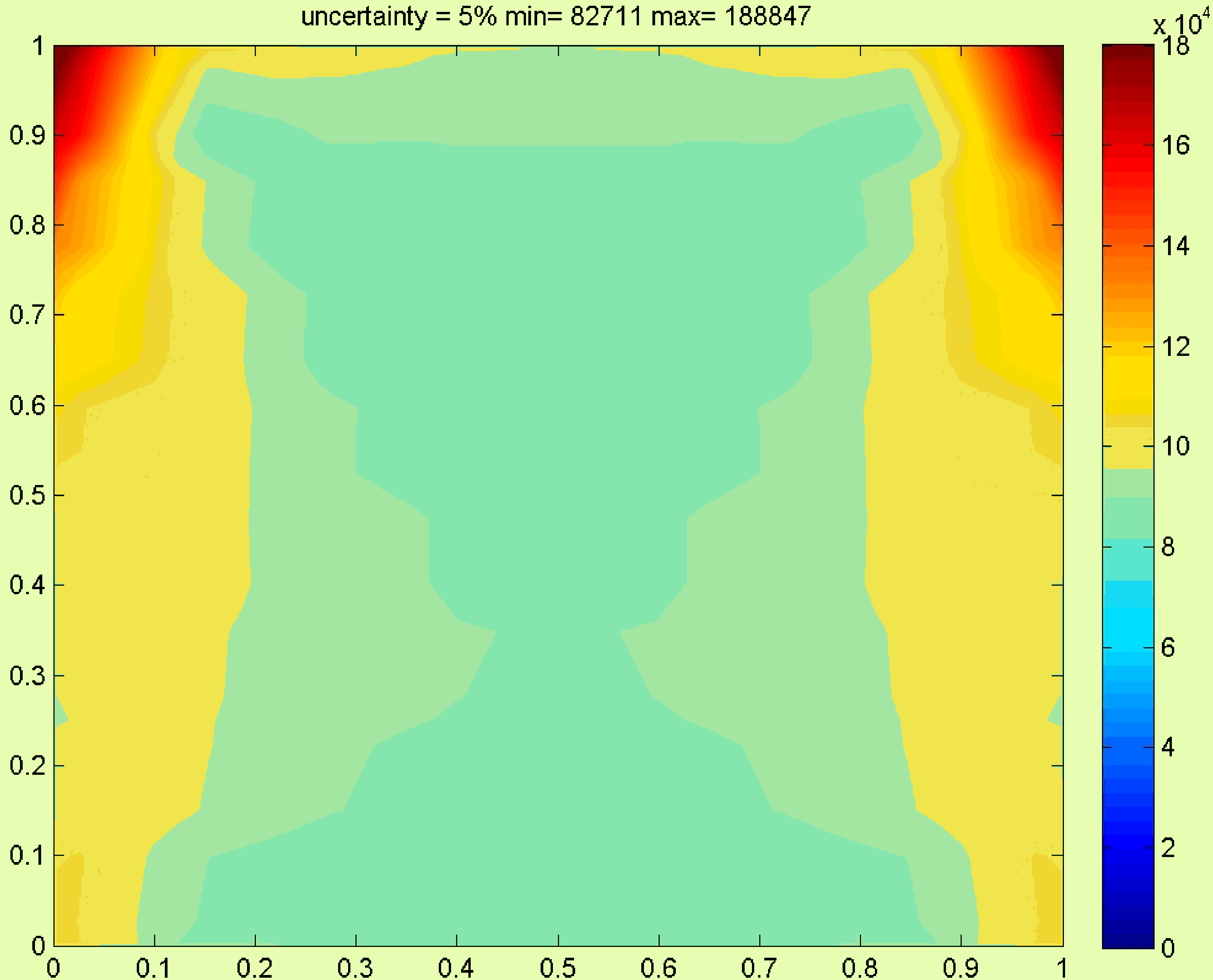
Interval solution

Table XV. Interval displacements 10% uncertainty

$\underline{u}$	$\overline{u}$	$\underline{u}$	$\overline{u}$		
0,1,1,0,1,0,1,1,1,0,1	1,0,0,1,0,1,$\boxed{1}$,0,0,1,0	0,1,1,0,1,0,1,1,1,0,1	1,0,0,1,0,1,$\boxed{0}$,0,0,1,0	0	1
0,0,0,0,1,0,1,1,1,0,1	1,1,1,$\boxed{0}$,0,1,0,0,0,1,0	0,0,0,0,1,0,1,1,1,0,1	1,1,1,$\boxed{1}$,0,1,0,0,0,1,0	0	1
0,0,1,0,1,0,1,1,1,0,1	1,$\boxed{0}$,0,1,0,1,0,0,0,1,0	0,$\boxed{1}$,1,0,1,0,1,1,1,0,1	1,0,0,1,0,1,0,0,0,1,0	1	0
0,0,1,1,0,1,0,0,0,0,1	1,1,0,1,1,0,1,1,1,1,0	0,0,1,1,0,1,0,0,0,0,1	1,1,0,1,1,0,1,1,1,1,0	0	0
0,0,1,1,0,1,0,0,1,0,1	1,1,0,0,1,0,1,1,$\boxed{1}$,1,0	0,0,1,1,0,1,0,0,$\boxed{0}$,0,1	1,1,0,0,1,0,1,1,1,1,0	1	0
0,0,0,0,1,0,1,1,1,1,0	1,1,1,1,0,$\boxed{0}$,0,0,0,0,1	0,0,0,0,1,0,1,1,1,1,0	1,1,1,1,0,$\boxed{1}$,0,0,0,0,1	0	1
0,0,1,1,0,1,0,0,0,0,1	1,$\boxed{0}$,0,0,1,0,1,1,1,1,0	0,0,1,1,0,1,0,0,0,0,1	1,$\boxed{1}$,0,0,1,0,1,1,1,1,0	0	1
1,0,0,1,0,1,0,0,0,1,0	0,1,1,0,1,1,1,1,1,0,1	1,0,0,1,0,1,0,0,0,1,0	0,1,1,0,1,1,1,1,1,0,1	0	0
0,0,1,0,1,1,0,1,0,0,1	1,$\boxed{0}$,0,$\boxed{0}$,0,0,1,0,1,1,0	0,$\boxed{1}$,1,$\boxed{1}$,1,1,0,1,0,0,1	1,0,0,0,0,0,1,0,1,1,0	2	0
0,0,0,0,0,0,1,1,1,0,1	1,1,1,1,1,1,0,0,0,1,0	0,0,0,0,0,0,1,1,1,0,1	1,1,1,1,1,1,0,0,0,1,0	0	0
0,1,1,1,1,1,0,0,0,0,1	1,0,0,0,0,0,1,1,1,1,0	0,1,1,1,1,1,0,0,0,0,1	1,0,0,0,0,0,1,1,1,1,0	0	0
0,0,1,0,0,1,0,0,0,0,1	1,1,0,1,1,0,1,1,1,1,0	0,0,1,0,0,1,0,0,0,0,1	1,1,0,1,1,0,1,1,1,1,0	0	0
0,0,1,0,0,0,1,0,1,0,1	1,$\boxed{0}$,0,1,1,$\boxed{0}$,0,1,0,1,0	0,0,1,0,0,0,1,0,1,0,1	1,$\boxed{1}$,0,1,1,$\boxed{1}$,0,1,0,1,0	0	2

2D problem with the interal parameters and interval functions parameters





uncertainty = 5% min= 82711 max= 188847
x 10^4
18
16
14
12
10
8
6
4
2
0
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Scripting language

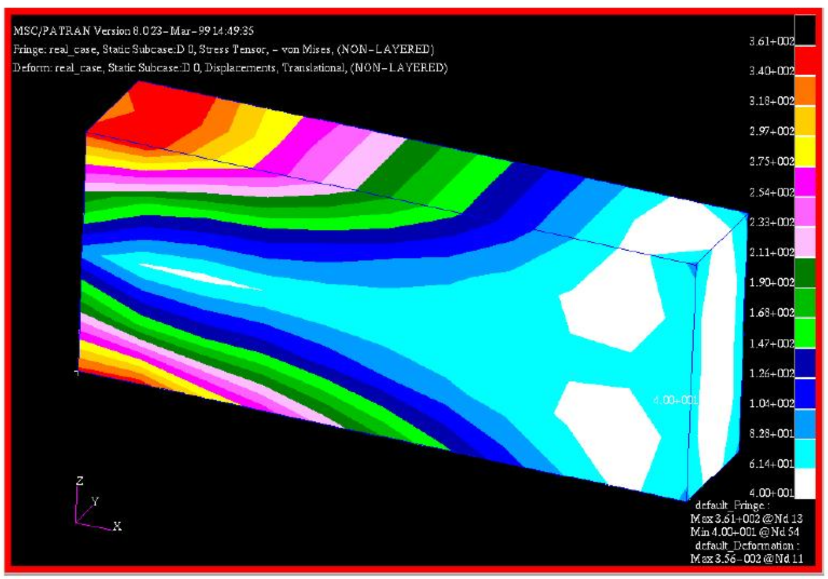
```
analysis_type linear_static_functional_derivative

parameter 1 [210E9,212E9]      # E
parameter 2 [0.2,0.4]          # Poisson number
parameter 3 0.1                # thickness
parameter 4 [-3,-1] sensitivity # fy

point 1 x 0 y 0
point 2 x 1 y 0
point 3 x 1 y 1
point 4 x 0 y 1
point 5 x -1 y 0
point 6 x -1 y 1

rectangle 1 points 1 2 3 4 parameters 1 2 3
rectangle 2 points 5 1 4 6 parameters 1 2 3

load constant_distributed_in_y_local_direction
geometrical_object 1 fy 4
load constant_distributed_in_y_local_direction
geometrical_object 2 fy 4

boundary_condition  fixed point 1 ux
boundary_condition  fixed point 1 uy
boundary_condition  fixed point 2 ux
boundary_condition  fixed point 2 uy
boundary_condition  fixed point 5 ux
boundary_condition  fixed point 5 uy

#Mesh
```

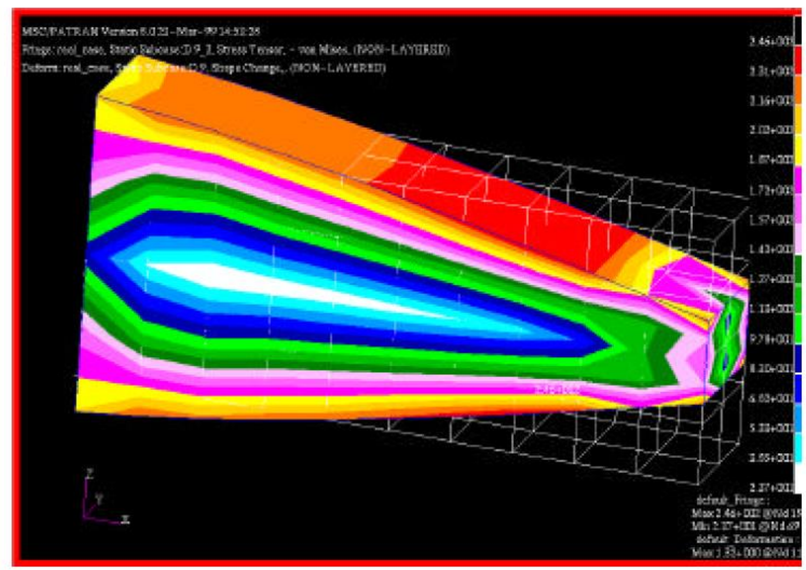
Shape optimization

$$\begin{cases} \min u = u(\Omega) \\ \Omega \in \left[\underline{\Omega}, \bar{\Omega}\right] \end{cases}$$

or

$$\begin{cases} \min u = u(\Omega) \\ C(\Omega) = 0 \\ \Omega \in \left[\underline{\Omega}, \bar{\Omega}\right] \end{cases}$$

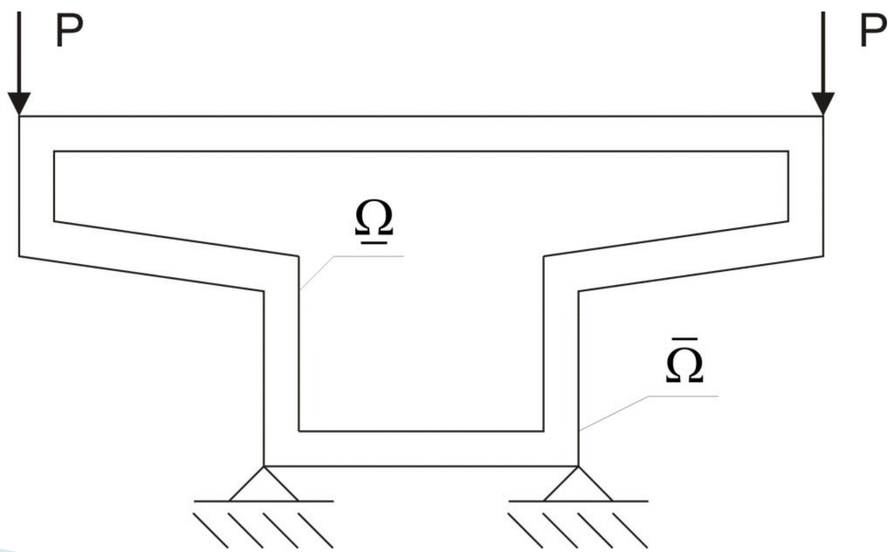




Interval set

$$A \le B \Leftrightarrow A \subseteq B$$

$$\mathbf{\Omega} = \left[\underline{\Omega}, \overline{\Omega} \right] = \left\{ \Omega : \underline{\Omega} \le \Omega \le \overline{\Omega} \right\}$$



Set dependent functions

Center of gravity

$$x_C\left(\Omega\right) = \frac{\int\limits_{\Omega} x\,d\mu}{\int\limits_{\Omega} d\mu}$$

Moment of inertia

$$I\left(\Omega\right) = \int\limits_{\Omega} y^2\,d\mu$$

Solution of PDE

$$\begin{cases} \Delta u + q = \dfrac{\partial u}{\partial t} \quad \text{for } \; x \in \operatorname{int}\Omega \\ u = u^* \quad \text{for } \; x \in \partial\Omega \end{cases}$$

Topological derivative

$$\lim_{\rho(\Delta\Omega)\to 0^+} \frac{u(\Omega+\Delta\Omega)-u(\Omega)}{f(\Delta\Omega)} = \lim_{\varepsilon\to 0} \frac{\frac{du}{d\varepsilon}}{\frac{df}{d\varepsilon}} = D_T(x)$$

Topological derivative

Examples

$$f(\Omega) = \int_{\Omega} L(x)\, d\mu \qquad \frac{df(\Omega)}{d\mu} = L(x)$$

$$x_C(\Omega) = \frac{\int_{\Omega} x\, d\mu}{\int_{\Omega} d\mu} \qquad \frac{dx_C(\Omega)}{d\mu} = \frac{x|\Omega| - \int_{\Omega} x\, d\mu}{\left(\int_{\Omega} d\mu\right)^2}$$

Extreme values of montone functions

$$f(x) = x^2, \quad x \in [1, 2]$$

$$\frac{df(x)}{dx} = 2x \in [2, 4]$$

$$\underline{f} = f\left(\underline{x}\right) = 1^2 = 1$$

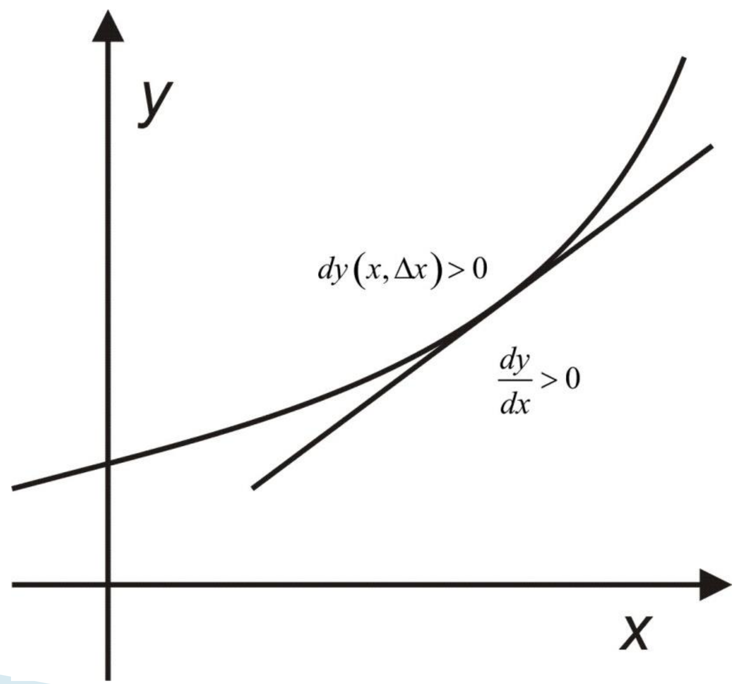
$$\overline{f} = f\left(\overline{x}\right) = 2^2 = 4$$

$$f(x) \in \left[\underline{f}, \overline{f}\right] = [1, 4]$$

Topological derivative and monotonicity

$$f\left(\Omega+\Delta\Omega\right)-f\left(\Omega\right)\approx df\left(\Omega,\Delta\Omega\right)\geq 0$$

$$f\left(\Omega+\Delta\Omega\right)\geq f\left(\Omega\right)$$



Parametric representation

$$\lim_{\varepsilon \to 0^+} \frac{u(\Omega_\varepsilon) - u(\Omega)}{f(\varepsilon)} = \lim_{\varepsilon \to 0} \frac{\frac{du}{d\varepsilon}}{\frac{df}{d\varepsilon}}$$

Results which are independent of parameterization

$$f(\Omega) = \left(\int_{\Omega} L(x)\, d\mu \right)^2$$

$$f(\Omega + \Delta\Omega) - f(\Omega) \approx 2 \left(\int_{\Omega} L(x)\, d\mu \right) L(x) \,|\, \Delta\Omega \,|$$

Results which are dependent on parameterization

$$\Omega = \left[x_0 , x_1 \right] \subset R$$

$$f(\Omega) = \frac{x - x_0}{x_1 - x_0} \qquad \left|\Omega\right| = \left|x_1 - x_0\right|$$

$$\frac{df}{d\mu} = \lim_{\Delta x_1 \to 0^+} \frac{f(x_1 + \Delta x_1) - f\left(x_1\right)}{\left|\Omega(x_1 + \Delta x_1)\right| - \left|\Omega(x_1)\right|} = \frac{\dfrac{df}{dx_1}}{\dfrac{d\Omega}{dx_1}} = -\frac{x - x_0}{\left(x_1 - x_0\right)^2}$$

Results which are dependent on parameterization

$$f(\Omega) = \frac{x - x_0}{x_1 - x_0} \qquad\qquad |\Omega| = |x_1 - x_0|$$

$$\frac{df}{d\mu} = \lim_{\Delta x_0 \to 0^+} \frac{f(x_0 + \Delta x_0) - f\left(x_0\right)}{|\Omega(x_0 + \Delta x_0)| - |\Omega(x_0)|} = \frac{\frac{df}{dx_0}}{\frac{d\Omega}{dx_0}} = \frac{x - x_1}{\left(x_1 - x_0\right)^2}$$

$$\frac{\frac{df}{dx_1}}{\frac{d\Omega}{dx_1}} \neq \frac{\frac{df}{dx_0}}{\frac{d\Omega}{dx_0}}$$

Computational method

Formulation of the problem

$$K(p)u = Q\left(p\right)$$

$$f\left(p\right) = \left|\Omega(p)\right|$$

Calculation of the derivative

$$K(p)\frac{\partial u}{\partial p} = \frac{\partial Q\left(p\right)}{\partial p} - \frac{\partial K\left(p\right)}{\partial p}u$$

$$\frac{\partial f\left(p\right)}{\partial p} = \frac{\partial \left|\Omega(p)\right|}{\partial p}$$

Computational method

$$\frac{\partial u}{\partial \Omega} = \frac{\dfrac{\partial u}{\partial p}}{\dfrac{\partial \Omega}{\partial p}}$$

If $du(\Omega, \Delta\Omega) \geq 0$ then $\Omega_{i+1} = \Omega_i + \Delta\Omega$

If $du(\Omega, \Delta\Omega) < 0$ then $\Omega_{i+1} = \Omega_i - \Delta\Omega$

$$\text{for } x \in \partial\Omega$$

Data for Calculations

- Coordinates of the nodes are the interval numbers.

$$x_i \in [\underline{x}_i, \overline{x}_i]$$

- Loads, material parameters (E, ν) can be also the interval numbers.

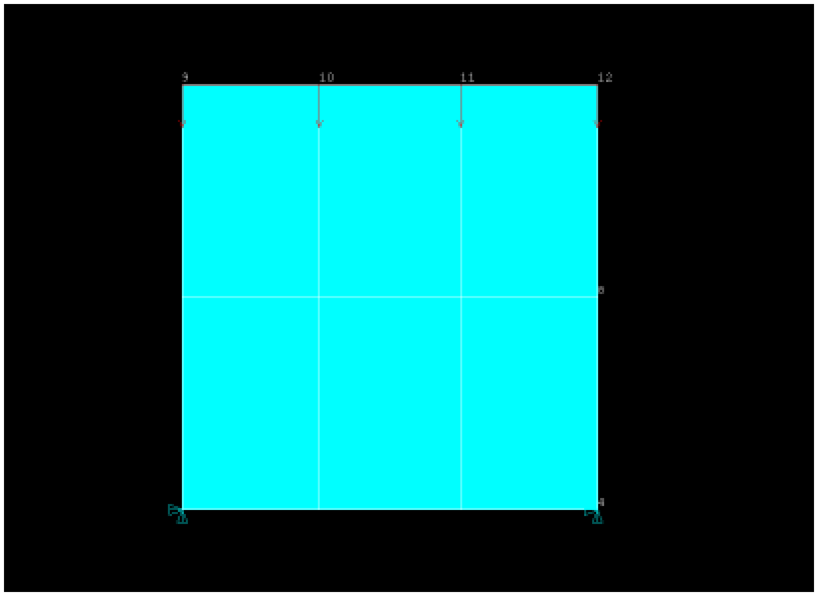
$$P \in [\underline{P}, \overline{P}], E \in [\underline{E}, \overline{E}], \nu \in [\underline{\nu}, \overline{\nu}]$$

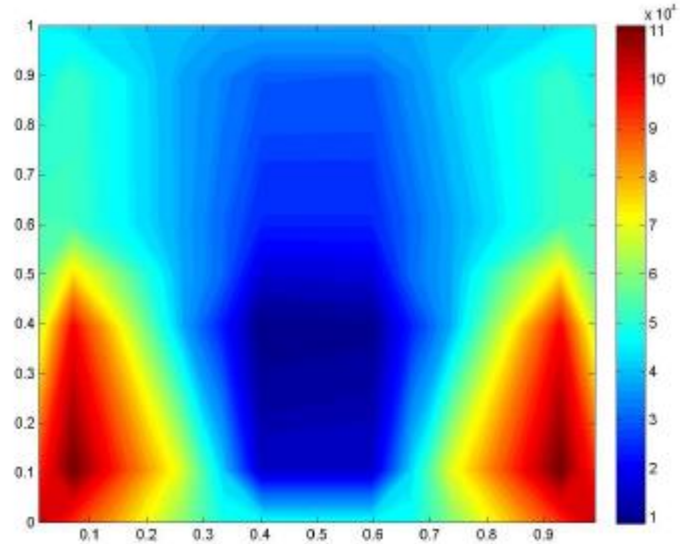
Interval solution

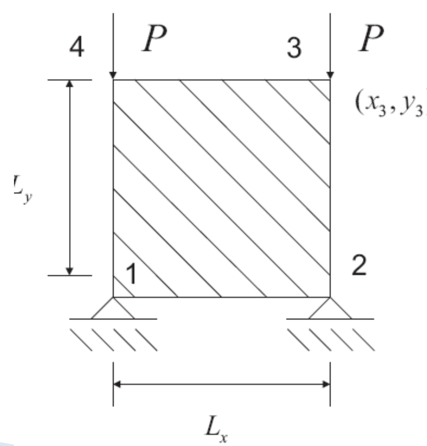
$$\mathbf{u}(x) = \lozenge\{u(x,\Omega) : \Omega \in [\underline{\Omega}, \overline{\Omega}]\}$$

$$\underline{u}(x) = u(x, \Omega^{min}),\ \overline{u}(x) = u(x, \Omega^{max})$$

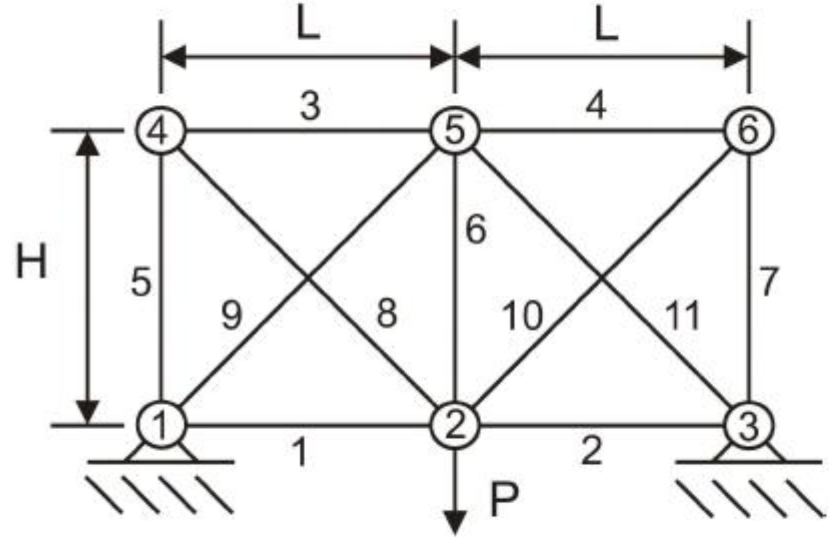
Interval von Mises stress







Truss structure with the uncertain geometry



	Lower bound [m]	Upper bound [m]
u_{2x}	-2.883822e-005	2.886680e-005
u_{2y}	-1.526831e-002	-1.463698e-002
u_{5x}	-7.216225e-005	7.214392e-005
u_{5y}	-1.296921e-002	-1.239865e-002

Conclusions

- Currend civil engineering codes are based on worst case design concept, because of that it is possible to use presented approach in the framework of existing law.

- Using presented method it is possible to solve engineering problems with interval parameters (interval parameters, functions, sets).