Application of order-preserving functions to the modeling of computational mechanics problems with uncertainty

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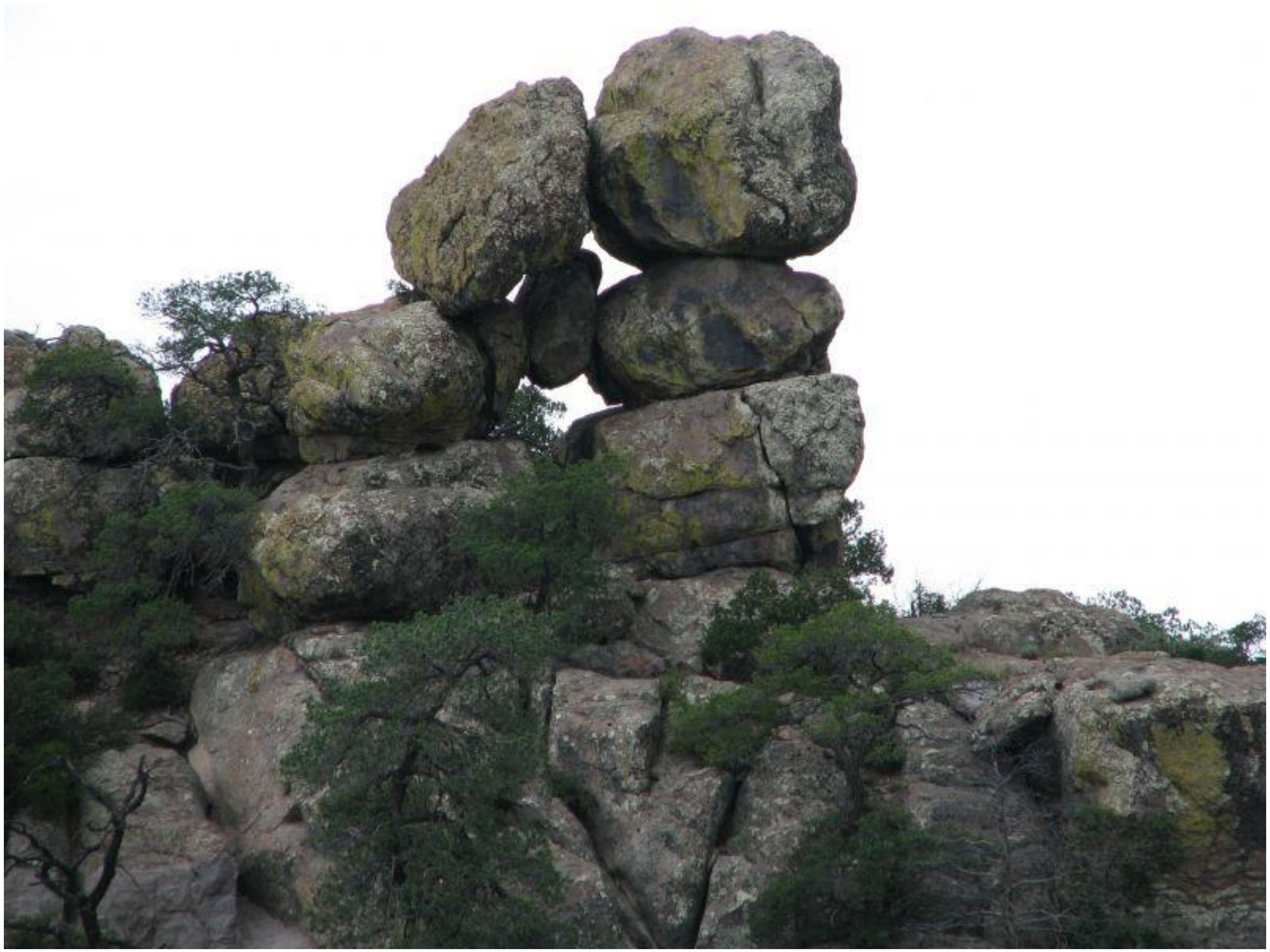
Agenda

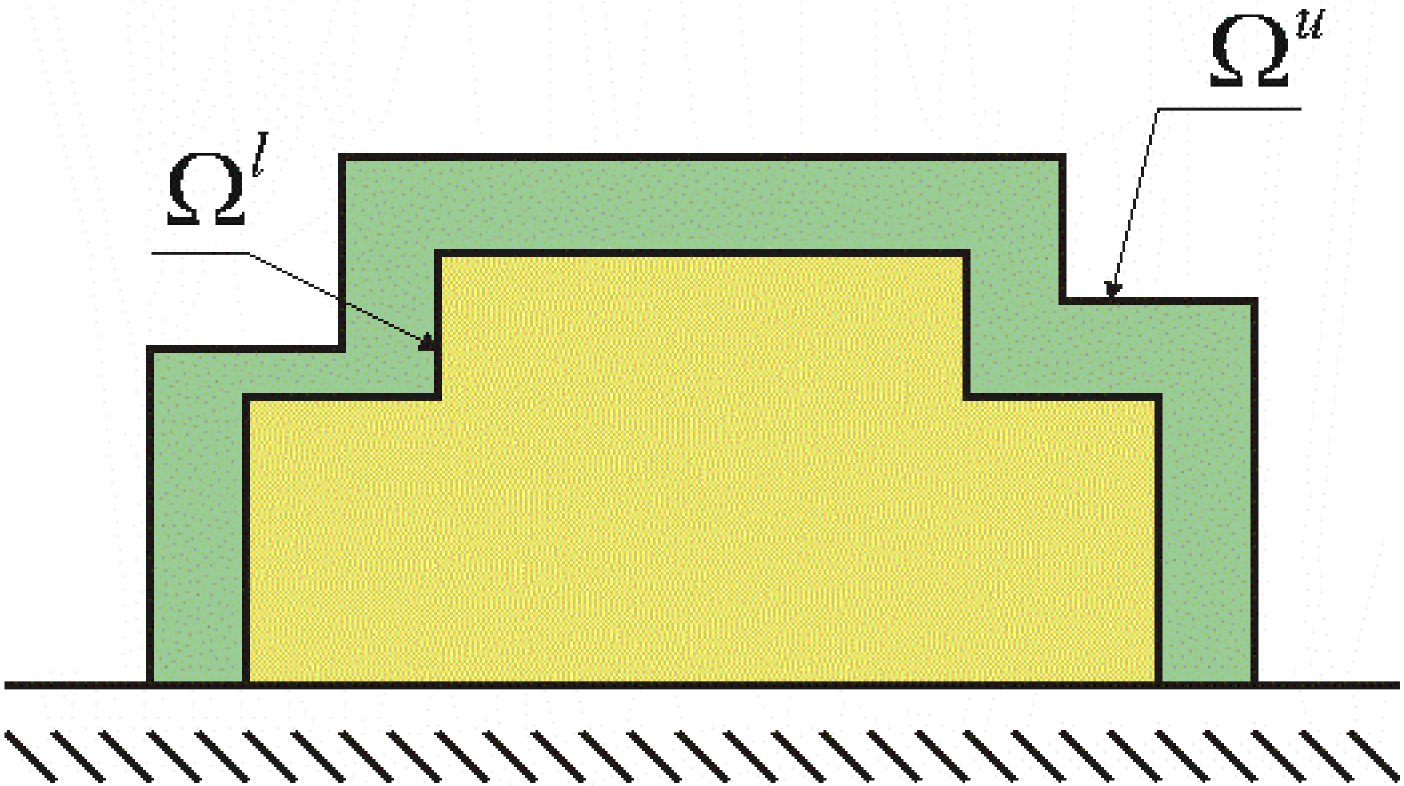
- Problems with uncertain geometry
- Modeling of uncertainty
- Sensitivity analysis
- Topological derivative
- Optimization based on topological derivative
- Examples of application
- Conclusion

Problems with uncertain geometry



Problems with uncertain geometry

Set valued uncertainty



$$\Omega \in \tilde{\Omega} = \left[\Omega^l, \Omega^u \right]$$

Set valued variables

$$f : R^n \supset M \ni x \rightarrow f(x) \in R$$

$$f(\Omega) = \{ f(x) : x \in \Omega \}$$

$$f^l = \inf f(\Omega)$$

$$f^u = \sup f(\Omega)$$

Monotone function

- if ($x<y$) then $f(x)>f(y)$
- if ($x<y$) then $f(x)<f(y)$

$$\left[x^l, x^u\right] = \{x \in X : x^l \le x \le x^u\}$$

- Interval numbers
- Interval functions
- Interval sets
- Interval ???

Integrals and the functions of integrals

$$f(\Omega) = \int_{\Omega} g(x)dx$$

$$f(\Omega) = F\left(\int_{\Omega} g(x)dx\right)$$

Examples

$$f(\Omega) = \int_{[0,L]} E(x)A(x)\frac{dN_1(x)}{dx}\frac{dN_2(x)}{dx}dx$$

$$f(\Omega) = \frac{\int_{\Omega} x\,d\Omega(x)}{\int_{\Omega} d\Omega(x)}$$

Numerical methods in engineering

$$K\left(\ldots,\int_{\Omega} g(x)d\Omega(x),\ldots\right)u = Q\left(\ldots,\int_{\Omega} g(x)d\Omega(x),\ldots\right)$$

$$u = u\left(\ldots,\int_{\Omega_e} g(x)d\Omega(x),\ldots\right) = u(\ldots,\Omega_e,\ldots)$$

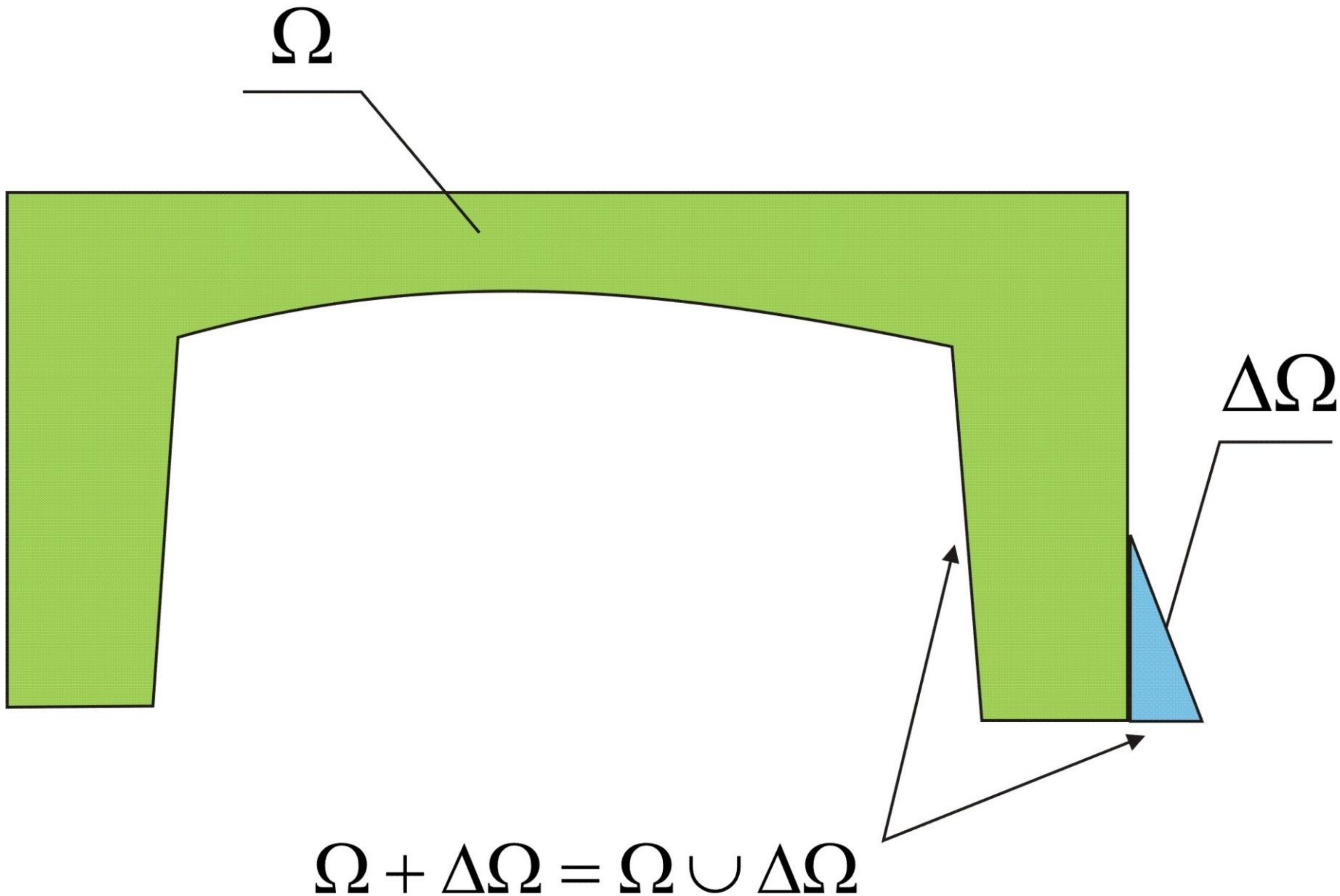
Displacement vector u is complicated function of many sets Ω_e

Optimization with set-valued variables

$$f^{l} = \inf\left\{ f(\tilde{\Omega}) : \tilde{\Omega} \in \left[\Omega^{l}, \Omega^{u}\right]\right\}$$

$$f^{u} = \sup\left\{ f(\tilde{\Omega}) : \tilde{\Omega} \in \left[\Omega^{l}, \Omega^{u}\right]\right\}$$

$$\left(A \leq B\right) \Leftrightarrow \left(A \subseteq B\right)$$


Ω
ΔΩ
Ω + ΔΩ = Ω ∪ ΔΩ

Differential of function with set-valued variables

$$f(\Omega + \Delta\Omega) - f(\Omega) \approx df(\Omega, \Delta\Omega)$$

For example

$$f(\Omega) = \int_{\Omega} g(x)\, d\Omega(x)$$

$$df(\Omega, \Delta\Omega) \approx g(x)\left|\Delta\Omega\right|$$

Example

$$I_z(\Omega) = \int_{\Omega} (x^2 + y^2) d\Omega$$

$$dI_z(x, y, \Omega, d\Omega) = (x^2 + y^2) \left| d\Omega \right|$$

Monotonicity

$$f(\Omega+\Delta\Omega)-f\left(\Omega\right)\geq 0\Rightarrow \boxed{f(\Omega+\Delta\Omega)\geq f\left(\Omega\right)}$$

$$f(\Omega+\Delta\Omega)-f(\Omega)\approx df(\Omega,\Delta\Omega)$$

If $df(\Omega,\Delta\Omega)\geq 0$ then

$$f(\Omega+\Delta\Omega)\geq f\left(\Omega\right)$$

Topological derivative

$$\lim_{|\Delta\Omega(x)|\to 0} \frac{f(\Omega + \Delta\Omega(x)) - f(\Omega)}{|\Delta\Omega(x)|} = \frac{df}{d\Omega(x)}$$

Topological derivative (example)

$$I_z(\Omega) = \int_{\Omega} (x^2 + y^2) d\Omega$$

$$dI_z(x, y, \Omega, d\Omega) = (x^2 + y^2)\left|d\Omega\right|$$

$$\frac{dI_z}{d\Omega(x, y)} = x^2 + y^2 \geq 0$$

Topological derivative (example)

$$\frac{dI_z}{d\Omega(x,y)} = x^2 + y^2 \geq 0$$

$$\Omega \subset \Omega + \Delta\Omega$$

$$I_z\left(\Omega + \Delta\Omega\right) > I_z\left(\Omega\right)$$

Differential of function with set-valued variables

$$f(\Omega) = F\left(\int_{\Omega} g(x)\, d\Omega(x)\right)$$

$$df(\Omega, \Delta\Omega) = F'\left(\int_{\Omega} g(x)\, d\Omega(x)\right) g(x)\left|\Delta\Omega\right|$$

Topological derivative

$$f(\Omega) = F\left(\int_{\Omega} g(x)\,d\Omega(x)\right)$$

$$\frac{df}{d\Omega(x)} = F'\left(\int_{\Omega} g(x)\,d\Omega(x)\right) g(x)$$

Example: center of gravity

$$f(\Omega) = \frac{\int_\Omega x d\Omega}{\int_\Omega d\Omega}$$

$$\frac{df(\Omega)}{d\Omega(x)} = \frac{\frac{d}{d\Omega(x)}\int_\Omega x d\Omega \cdot \int_\Omega d\Omega - \int_\Omega x d\Omega \cdot \frac{d}{d\Omega(x)}\int_\Omega d\Omega}{\left(\int_\Omega d\Omega\right)^2}$$

Example: center of gravity

$$\frac{df(\Omega)}{d\Omega(x)} = \frac{x\int_{\Omega} d\Omega - \int_{\Omega} x d\Omega}{\left(\int_{\Omega} d\Omega\right)^2}$$

System of algebraic equations

$$K(\Omega)u(\Omega) = Q\left(\Omega\right)$$

$$dK(\Omega, d\Omega)\cdot u(\Omega) + K(\Omega)\cdot du(\Omega, d\Omega) = dQ\left(\Omega, d\Omega\right)$$

$$du(\Omega, d\Omega)$$

can be used in optimization problems.

System of algebraic equations

$$K(\Omega)u(\Omega) = Q\left(\Omega\right)$$

$$K(\Omega)\cdot du(\Omega, d\Omega) = dQ\left(\Omega, d\Omega\right) - dK(\Omega, d\Omega)\cdot u(\Omega)$$

$$\left.\begin{array}{l} K(\Omega) \\ dQ\left(\Omega, d\Omega\right) \\ dK\left(\Omega, d\Omega\right) \end{array}\right\}$$ – Can be calculated directly

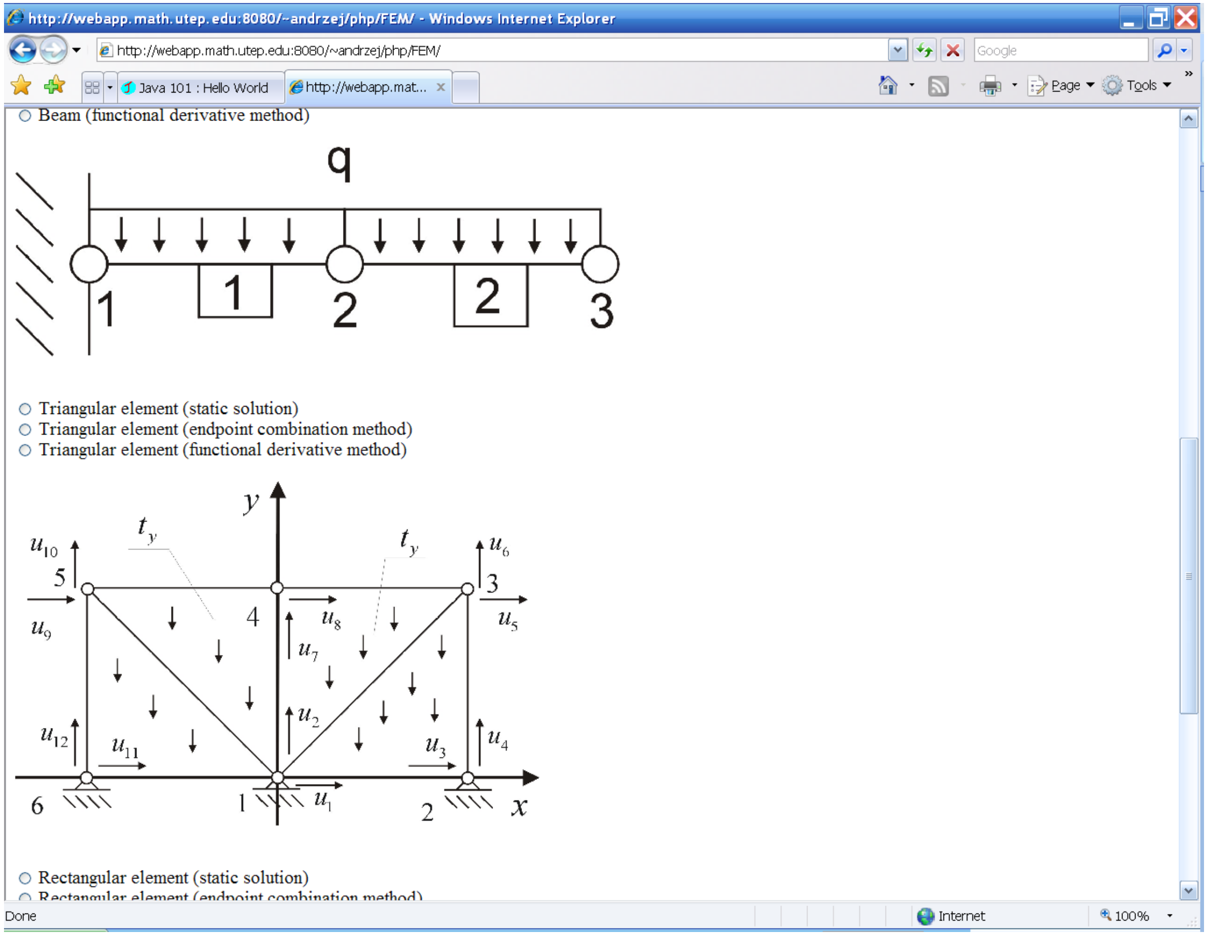
$u(\Omega), du(\Omega, d\Omega)$ – Solution of the system of equation

Topological derivative

$$K(\Omega) \cdot \frac{du(\Omega)}{d\Omega(x)} = \frac{dQ(\Omega)}{d\Omega(x)} - \frac{dK(\Omega)}{d\Omega(x)} \cdot u(\Omega)$$

Web application

http://andrzej.pownuk.com/



Differential equations

$$F\left(x, y(x,\Omega), \frac{dy(x,\Omega)}{dx}, \Omega\right) = 0$$

$$\frac{\partial F}{\partial y} dy(x, d\Omega) + \frac{\partial F}{\partial \mathrm{v}} \frac{d}{dx} dy(x, d\Omega) + dF_{\Omega}(x, y, \mathrm{v}, d\Omega) = 0$$

where $\mathrm{v} = \frac{dy(x,\Omega)}{dx}$.

From this equation it is possible to calculate *d*y and then use this result in the sensitivity analysis.

Differential equations

$$F\left(x, y(x,\Omega), \frac{dy(x,\Omega)}{dx}, \Omega \right) = 0$$

$$\frac{\partial F}{\partial y}\frac{dy}{d\Omega(x)} + \frac{\partial F}{\partial \mathrm{v}}\frac{d}{dx}\left(\frac{dy}{d\Omega(x)} \right) + \frac{\partial F}{\partial \Omega(x)} = 0$$

Using presented approach it is possible to avoid approximations errors.

Linear elasticity

$$\sum_j \mu \frac{\partial^2 u_i}{\partial x_j^2} + \sum_j (\mu+\lambda)\frac{\partial^2 u_j}{\partial x_i \partial x_j} + F_i = \rho \frac{\partial^2 u_i}{\partial t^2}$$

$$\sum_j \mu \frac{\partial^2}{\partial x_j^2}\left(\frac{\partial u_i(x,t)}{\partial \Omega(y)}\right) + \sum_j (\mu+\lambda)\frac{\partial^2}{\partial x_i \partial x_j}\left(\frac{\partial u_j(x,t)}{\partial \Omega(y)}\right) = \rho \frac{\partial^2}{\partial t^2}\left(\frac{\partial u_i(x,t)}{\partial \Omega(y)}\right)$$

$$\sum_j \mu \frac{\partial^2 \psi_i}{\partial x_j^2} + \sum_j (\mu+\lambda)\frac{\partial^2 \psi_j}{\partial x_i \partial x_j} = \rho \frac{\partial^2 \psi_i}{\partial t^2}$$

$$\psi_j = \frac{\partial u_j(x,t)}{\partial \Omega(y)}$$

Tension-compression example

$$\frac{d}{dx}\left(E\left|A\right|\frac{du(x,A)}{dx} \right) + n = 0$$

$$\frac{\partial}{\partial x}\left(E\frac{\partial}{\partial x}u(x,A) + E\left|A\right|\frac{\partial}{\partial x}\left(\frac{\partial u(x,A)}{\partial A(\xi)} \right) \right) = 0$$

Direct method

$$u(x,A)=\int_0^L \frac{P}{E\left|A\right|}dx$$

$$\left|A\right|=\int_A dA(\xi)$$

$$\frac{\partial}{\partial A(\xi)}\frac{1}{|A|}=-\frac{1}{\left|A\right|^2}\frac{\partial}{\partial A(\xi)}|A|=-\frac{1}{\left|A\right|^2}1=-\frac{1}{\left|A\right|^2}$$

Upper and lower bound

$$\frac{\partial u(x,A)}{\partial A(\xi)} = \frac{\partial}{\partial A(\xi)} \int_0^L \frac{P}{E|A|} dx = -\int_0^L \frac{P}{E|A|^2} dx < 0$$

$$u^l = u(A^u)$$

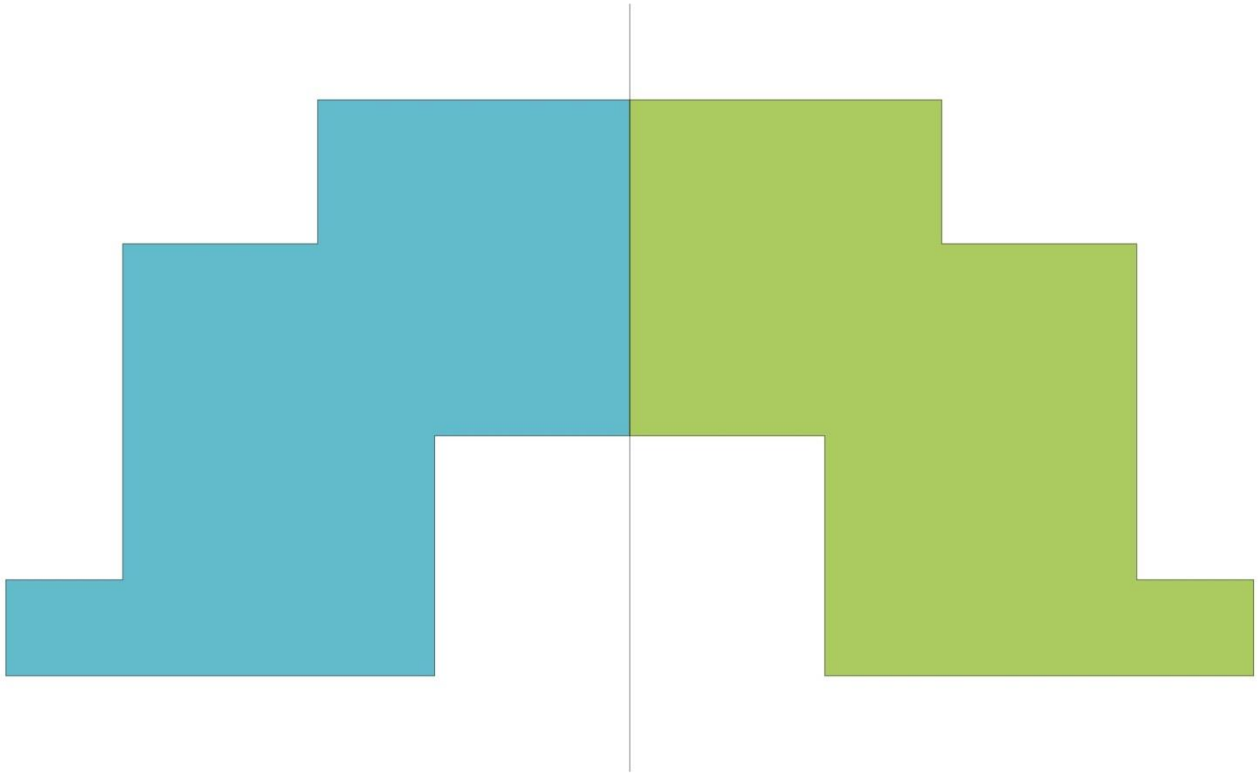
$$u^u = u(A^l)$$

Center of gravity

$$\frac{df(\Omega)}{d\Omega(x)} = \frac{x\int\limits_{\Omega} d\Omega - \int\limits_{\Omega} x d\Omega}{\left(\int\limits_{\Omega} d\Omega\right)^2} = 0$$

$$x = \frac{\int\limits_{\Omega} x d\Omega}{\int\limits_{\Omega} d\Omega}$$

Extreme values of the solution



$$x\int_{\Omega} d\Omega - \int_{\Omega} x\,d\Omega < 0 \qquad x_C \qquad x\int_{\Omega} d\Omega - \int_{\Omega} x\,d\Omega > 0$$

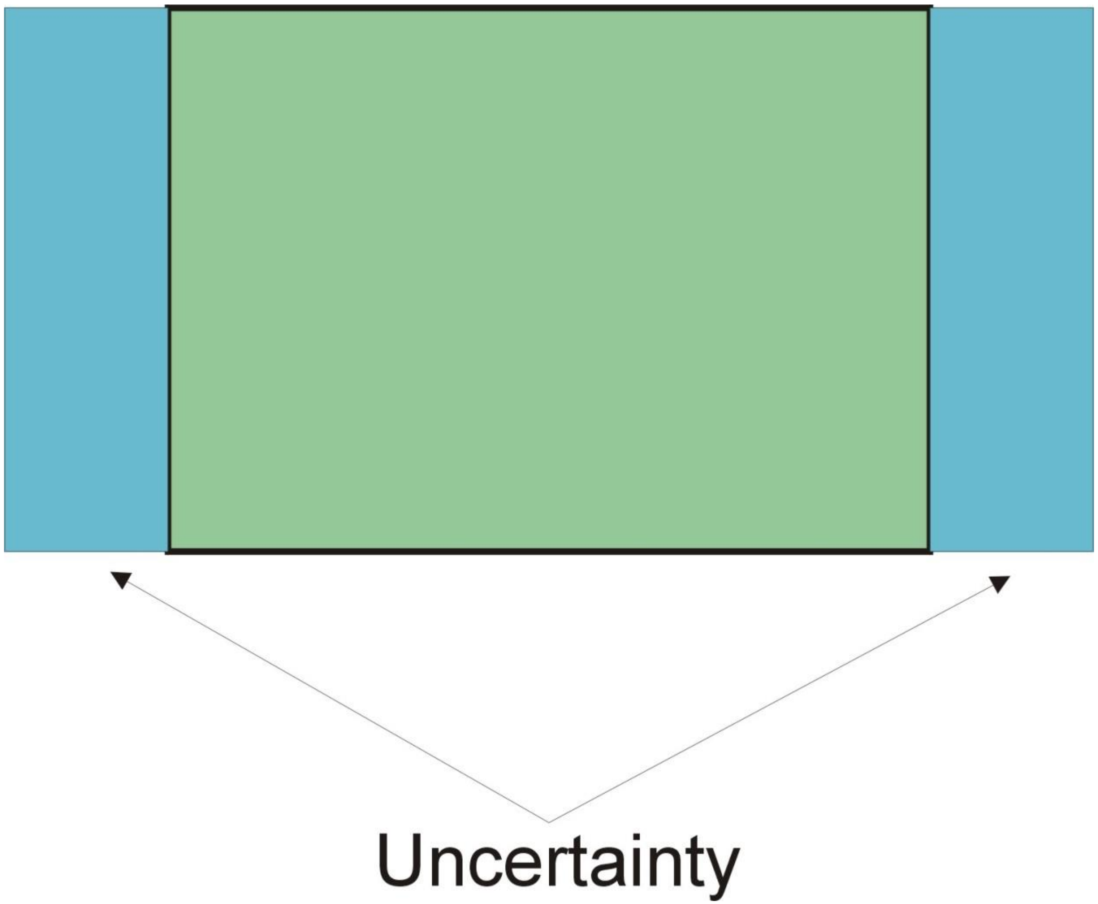
Interesting property

$$g(x,\Omega) = x\int_{\Omega} d\Omega - \int_{\Omega} x d\Omega$$

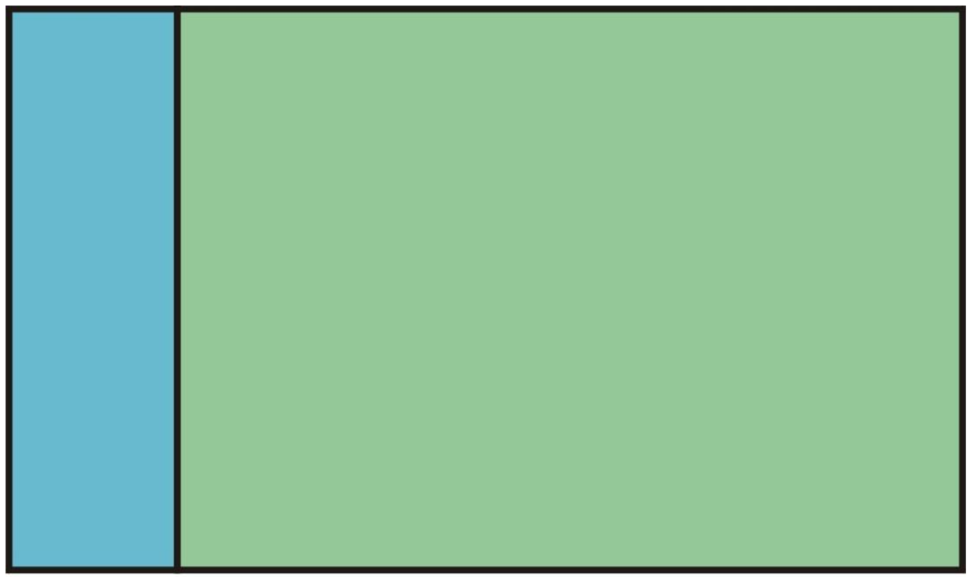
$$\frac{dg(\Omega)}{d\Omega(x)} = \frac{d}{d\Omega\left(x\right)}\left[x\int_{\Omega} d\Omega - \int_{\Omega} x d\Omega\right] =$$

$$= x \cdot 1 - x = 0$$

Center of gravity

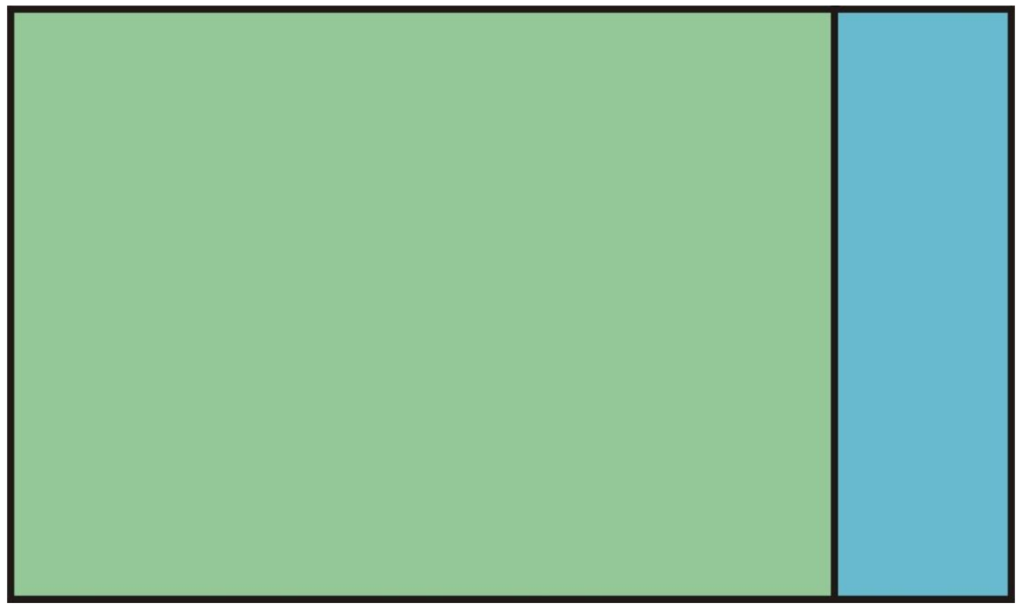


Center of gravity



$$x_c^l = x_C(\Omega^{\min})$$

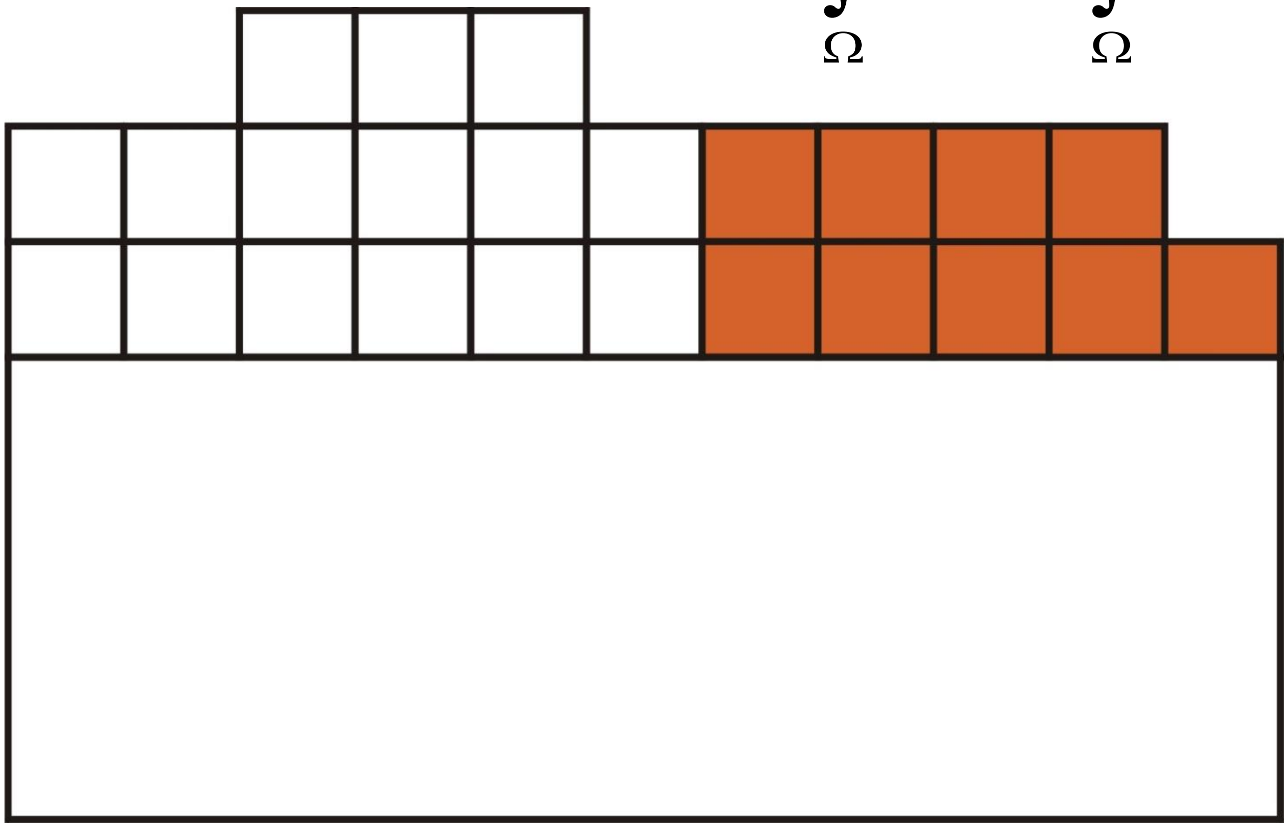
Center of gravity



$$x_c^u = x_C(\Omega^{\max})$$

Iterative methods.

$$x\int_{\Omega} d\Omega - \int_{\Omega} x d\Omega < 0$$



Numerical examples use code which is written in Java

Iterative method (lower bound)

1) Create a list of boxes $(\Delta\Omega_i, x_i)$

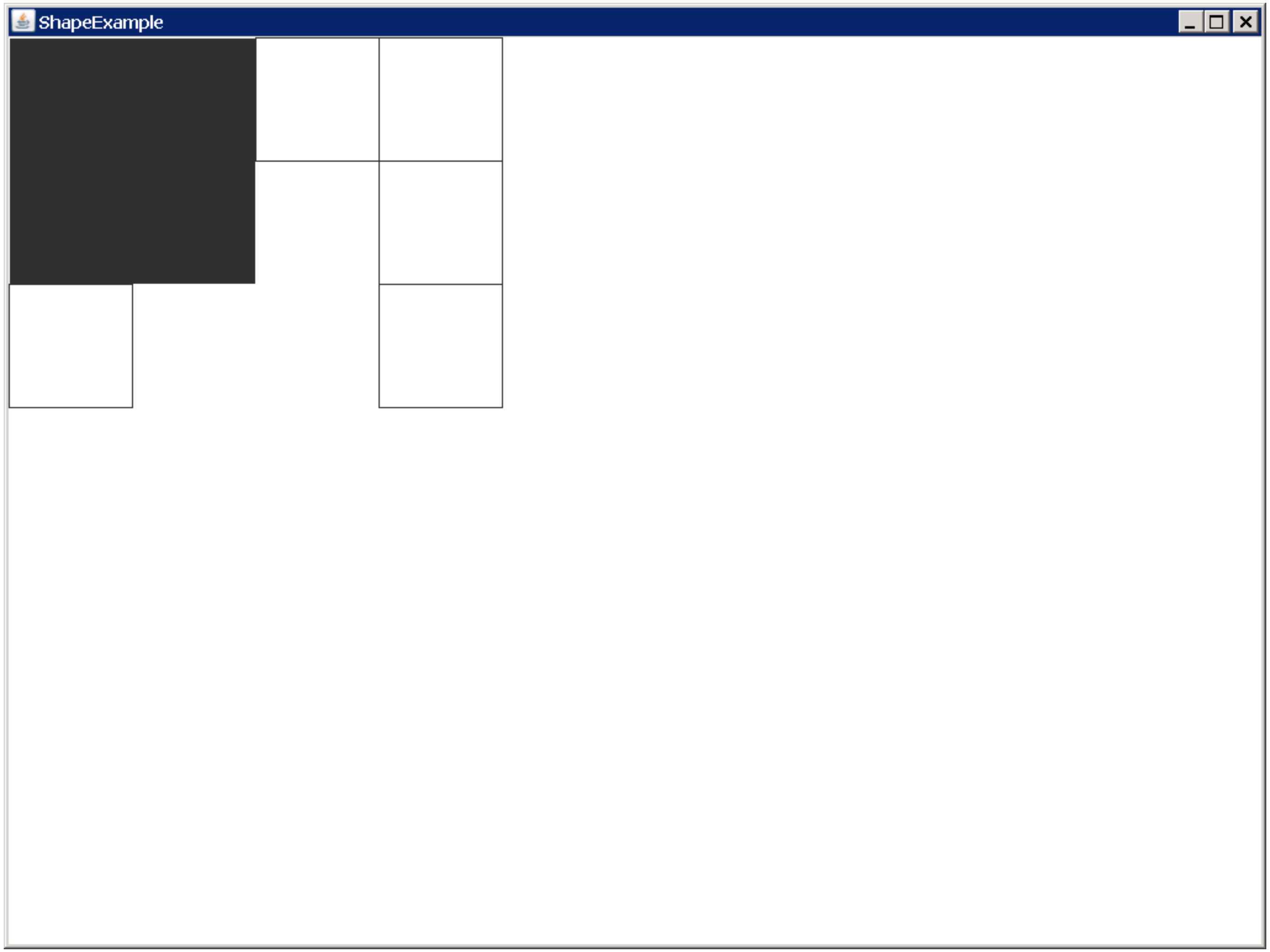
2) Calculate the area $\Omega = \bigcup_i \Delta\Omega_i$.

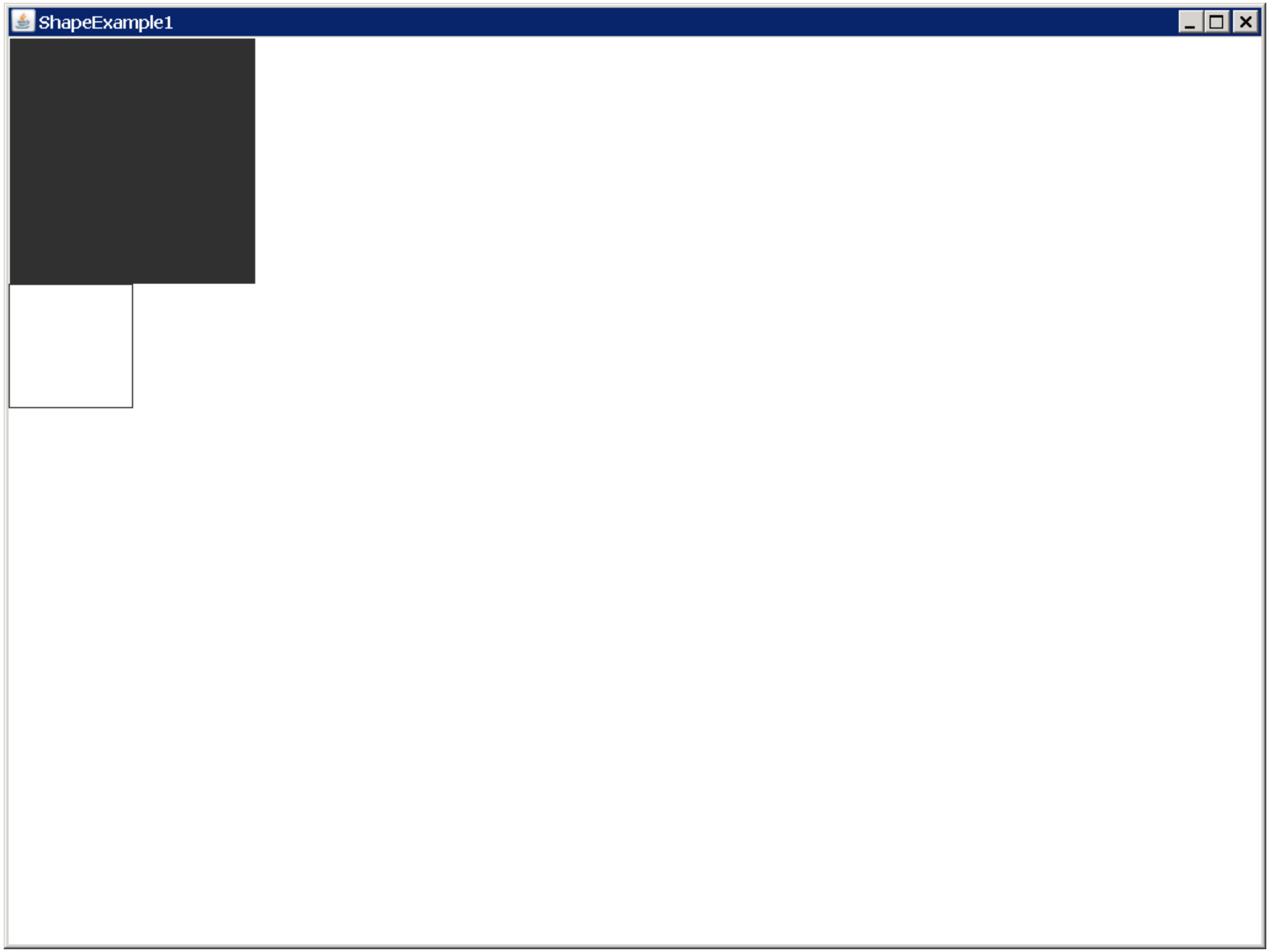
3) If $\dfrac{dx_C(x_i,\Omega)}{d\Omega(x)} \geq 0$ remove the box form the list.

4) Calculate new center of gravity x_C^{new}.

5) If $\left| x_C^{new} - x_C^{old} \right| < \varepsilon$ then, stop.

6) Go to point 2.


ShapeExample


ShapeExample1

F:\WINDOWS\system32\cmd.exe - java ShapeExample

```
number 0 area= 9801.0 xc= 50.5 is uncertain 0 is a member 1
number 1 area= 9900.0 xc= 50.5 is uncertain 0 is a member 1
number 2 area= 9900.0 xc= 150.0 is uncertain 0 is a member 1
number 3 area= 10000.0 xc= 150.0 is uncertain 0 is a member 1
number 4 area= 10000.0 xc= 250.0 is uncertain 1 is a member 1
number 5 area= 10000.0 xc= 350.0 is uncertain 1 is a member 1
number 6 area= 10000.0 xc= 350.0 is uncertain 1 is a member 1
number 7 area= 10000.0 xc= 350.0 is uncertain 1 is a member 1
number 8 area= 10000.0 xc= 50.0 is uncertain 1 is a member 1
 sensitivity (minus) [4] = 6.128726473573921E-4
current xc (before)195.0859979241303
rectangle 4 was subtracted
current xc (after)188.18734061129885
 sensitivity (minus) [5] = 0.002032796816480963
current xc (before)188.18734061129885
rectangle 5 was subtracted
current xc (after)164.9387293286016
 sensitivity (minus) [6] = 0.00265888100334742
current xc (before)164.9387293286016
rectangle 6 was subtracted
current xc (after)133.88870153185348
 sensitivity (minus) [7] = 0.003625967659404146
current xc (before)133.88870153185348
rectangle 7 was subtracted
current xc (after)90.3187536541602
 sensitivity (minus) [8] = -8.128617095252152E-4
 sensitivity (plus) [4] = 0.0032193150611044092
 sensitivity (plus) [5] = 0.0052354034464192215
 sensitivity (plus) [6] = 0.0052354034464192215
 sensitivity (plus) [7] = 0.0052354034464192215
 xcMin = 195.0859979241303 xcMinNew 90.3187536541602
 sensitivity (minus) [8] = -8.128617095252152E-4
 sensitivity (plus) [4] = 0.0032193150611044092
 sensitivity (plus) [5] = 0.0052354034464192215
 sensitivity (plus) [6] = 0.0052354034464192215
 sensitivity (plus) [7] = 0.0052354034464192215
 xcMin = 90.3187536541602 xcMinNew 90.3187536541602
 xcMin = 90.3187536541602
number 0 area= 9801.0 xc= 50.5 is uncertain 0 is a member 1
number 1 area= 9900.0 xc= 50.5 is uncertain 0 is a member 1
number 2 area= 9900.0 xc= 150.0 is uncertain 0 is a member 1
```

Work done on uncertain path

$$W = W(L) = \int_L P dx + Q dy = \int_L \omega$$

$$L \in [L^l, L^u]$$

$$L^{\min} = \left\{ x \in L^{uncertain} : dW(x, dL) < 0 \right\}$$

$$L^{\max} = \left\{ x \in L^{uncertain} : dW(x, dL) \geq 0 \right\}$$

Work done on uncertain path

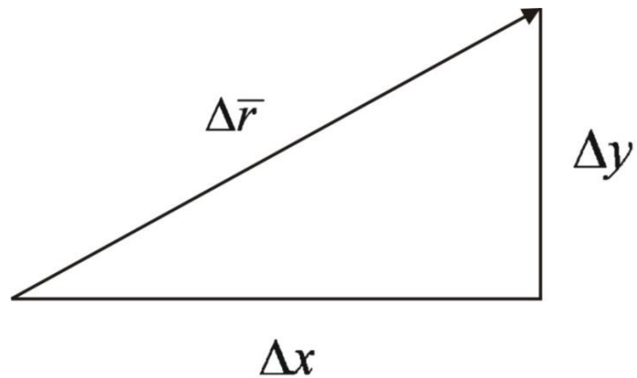
$$dW(x, dL) = \omega(d\overline{r}) = Pdx + Qdy$$

$$W^{l} = W(L^{\min})$$

$$W^{u} = W(L^{\max})$$

Work done on uncertain path

$$dL = \left| d\overline{r} \right|$$



$$dW(x, \Delta L) \approx P\Delta x + Q\Delta y$$

$$dW(x, \Delta\overline{r}) = \omega(\Delta\overline{r}) =$$

$$= P dx(\Delta\overline{r}) + Q dy(\Delta\overline{r}) = P\Delta x + Q\Delta y$$

Topological derivative

$$dW(x, \Delta\bar{r}) = \omega(\Delta\bar{r}) =$$

$$= Pdx(\Delta\bar{r}) + Qdy(\Delta\bar{r}) = P\frac{dx}{dt}\Delta t + Q\frac{dy}{dt}\Delta t$$

$$\frac{dW}{dL(t)} = P\frac{dx}{dt} + Q\frac{dy}{dt}$$

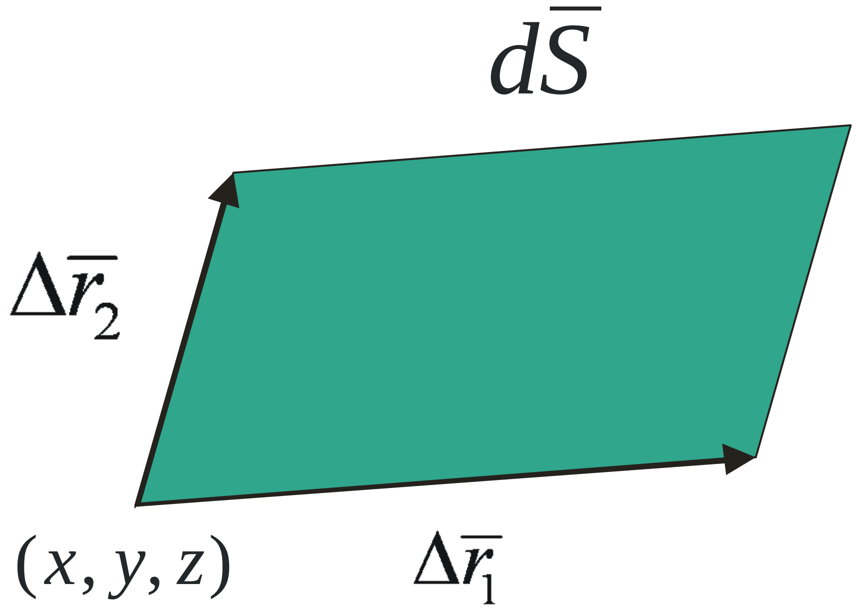
Flux integral

$$f(S) = \iint_S P dy \wedge dz + Q dz \wedge dx + R dx \wedge dy = \iint_S \omega$$

$$df(\Delta \overline{S}) = \omega(\Delta \overline{S}) = \omega(\Delta \overline{r}_1, \Delta \overline{r}_2)$$

$$\omega = P dy \wedge dz + Q dz \wedge dx + R dx \wedge dy$$

Geometrical interpretation of ds



$$\omega(\Delta\overline{S}) = \omega(\Delta\overline{r}_1, \Delta\overline{r}_2)$$

Sensitivity analysis

- Divide the S into $\Delta \bar{S}_i$.
- Check the sign of $\omega(\Delta \bar{S}_i)$.
- Depending on the sign of $\omega(\Delta \bar{S}_i)$ define $S^{\min}, S^{\max}$.
- $f^{l} = f(S^{\min}), f^{u} = f(S^{\max})$.

Topological derivative of the flux integral

$$\omega = Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy = \left(P\frac{\partial(y,z)}{\partial(s,t)} + Q\frac{\partial(z,x)}{\partial(s,t)} + R\frac{\partial(y,z)}{\partial(s,t)} \right) ds \wedge dt$$

$$\frac{d\omega}{dS(s,t)} = P\frac{\partial(y,z)}{\partial(s,t)} + Q\frac{\partial(z,x)}{\partial(s,t)} + R\frac{\partial(y,z)}{\partial(s,t)}$$

Line integral

$$f(L) = \int_L f dl = \int_L f \sqrt{dx^2 + dy^2 + dz^2} = \int_L \omega$$

$$df(d\overline{r}) = f \sqrt{dx^2 + dy^2 + dz^2}$$

$$\frac{df}{dL(t)} = f \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

Surface integral

$$f(S)=\int_L fdS=\int_L f\sqrt{\left(dy\wedge dz\right)^2+\left(dz\wedge dx\right)^2+\left(dx\wedge dy\right)^2}=\int_S \omega$$

$$df(S,d\overline{S})=f\sqrt{\left(dy\wedge dz\right)^2+\left(dz\wedge dx\right)^2+\left(dx\wedge dy\right)^2}$$

$$\frac{df}{dS\left(s,t\right)}=f\sqrt{\left(\frac{\partial(y,z)}{\partial(s,t)}\right)^2+\left(\frac{\partial(z,x)}{\partial(s,t)}\right)^2+\left(\frac{\partial(y,z)}{\partial(s,t)}\right)^2}$$

General integral form differential form

$$f(M) = \int_{M} \omega$$

$$df(d\bar{M}) = \omega\left(d\bar{M}\right)$$

Example (volume integral)

- Let us consider volume integral

$$f(V) = \int_V g\,dx \wedge dy \wedge dz$$

- If the volume is regular enough then

$$f(V) = \left(\int_{D_{xy}} \left(\int_0^{z(x,y)} g\,dz \right) dxdy \right)$$

Example (volume integral)

$$V = \{(x, y, z) : 0 \le z \le z(x, y), (x, y) \in D_{xy}\}$$

$$df(dV) \approx \int_0^{z(x,y)} g dz \left|\Delta D_{xy}\right| + \iint_{D_{xy}} g \delta z dx dy + g \delta z \left|\Delta\Omega\right|$$


7: Welded Tuff

Acknowledgements

- I would like to thank
  Dr Behzad Rouhani
  and Dr Vladik Kreinovich
  for their useful comments.

Conclusions

- Sensitivity analysis can be applied to the solution of problems with uncertain geometry.
- If it is not possible to apply topological derivative then the sign of differential can be used.