Solution of the Interval Equations of Dynamics by Using Adaptive Approximation

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***Abstract*—In this paper a new method for solution of the interval equations of dynamics will be presented. In this approach, in order to estimate upper ($\overline{y}$) and lower bound ($\underline{y}$) of the solution $y = y(t,p)$ it is necessary to create approximation $y^{approx} = y^{approx}(t, p_0, ..., p_n, p)$ ($p_0, ..., p_n$ are some point values, and $y \approx y^{approx}$). Then this approximation can be applied for calculation of $\underline{y}^{approx}$ and $\overline{y}^{approx}$. It is also possible to get reliable inner estimation $\underline{y}^{inner}$ and $\overline{y}^{inner}$ of the solution. Using the differences $\underline{y}^{approx} - \underline{y}^{inner}$, $\overline{y}^{approx} - \overline{y}^{inner}$ it is possible to control accuracy of the calculations. This method gives the possibility to calculate combinations of parameters ($p^{min}_{approx}(t), p^{max}_{approx}(t), p^{min}_{inner}(t), p^{max}_{inner}(t)$), which generate both interval solutions (i.e. $\underline{y}^{approx} = y(t, p^{min}_{approx}(t))$, $\overline{y}^{approx} = y(t, p^{max}_{approx}(t))$, $\underline{y}^{inner} = y(t, p^{min}_{inner}(t))$, $\overline{y}^{inner} = y(t, p^{max}_{inner}(t))$). This is very useful in applications of the interval methods.**

I. Solution set of the interval equations

Let us consider equation

$$F(y, t, p) = Q(t, p) \tag{1}$$

(algebraic, differential, integral, etc.) where $x \in R^n, p \in \mathbf{p} \subset R^m$. Solution set of the equation (1) can be defined in the following way [1], [5]

$$\mathbf{y}(t) = \{y(t,p) : F(y,t,p) = Q(t,p), p \in \mathbf{p}\} \tag{2}$$

$$\underline{y}(t) = min\ \mathbf{y}(t), \overline{y}(t) = max\ \mathbf{y}(t) \tag{3}$$

Interval solution may be also defined as a solution of the following optimization problems

$$\underline{y}(t) = \begin{cases} min\ y(t,p) \\ F(y,t,p) = Q(t,p) \\ p \in \mathbf{p} \end{cases} \tag{4}$$

$$\overline{y}(t) = \begin{cases} max\ y(t,p) \\ F(y,t,p) = Q(t,p) \\ p \in \mathbf{p} \end{cases} \tag{5}$$

$$\overline{p}^{min}(t) = argmin \begin{cases} min\ y(t,p) \\ F(y,t,p) = Q(t,p) \\ p \in \mathbf{p} \end{cases} \tag{6}$$

$$\overline{p}^{max}(t) = argmax \begin{cases} max\ y(t,p) \\ F(y,t,p) = Q(t,p) \\ p \in \mathbf{p} \end{cases} \tag{7}$$

$$\underline{y}(t) = y(t, p^{min}(t)),\ \overline{y}(t) = y(t, p^{max}(t)) \tag{8}$$

Presented definition was applied to the solution of many equations with the interval parameters [1], [2], [3], [4].

II. The use of monotonicity

A. Free vibrations

Let us consider the following differential equation

$$\frac{d^2y}{dt^2} + \omega^2 y = 0 \tag{9}$$

where $\omega^2 = \frac{k}{m}$. Let's assume that $y(0) \in [1, 2]$, $\frac{dy(0)}{dx} \in [1, 2]$ and $\omega = 1$. In this case we have two interval parameters $p_1 = y(0)$ and $p_2 = \frac{dy(0)}{dt}$ additionally derivative with respect to each uncertain parameter $\frac{\partial y}{\partial p_1}$, $\frac{\partial y}{\partial p_2}$ have constant sign. Because of that it is possible to calculate the solution exactly by using appropriate endpoints of the parameters.

$$\underline{y}(t) = \begin{cases} 1\cos(t) + 1\sin(t), t \in \left[0, \frac{\pi}{2}\right] \\ 2\cos(t) + 1\sin(t), t \in \left[\frac{\pi}{2}, \pi\right] \\ 2\cos(t) + 2\sin(t), t \in \left[\pi, \frac{3\pi}{2}\right] \\ 1\cos(t) + 2\sin(t), t \in \left[\frac{3\pi}{2}, 2\pi\right] \end{cases} \tag{10}$$

$$\bar{y}(t) = \begin{cases} 2\cos(t) + 2\sin(t), t \in \left[0, \frac{\pi}{2}\right] \\ 1\cos(t) + 2\sin(t), t \in \left[\frac{\pi}{2}, \pi\right] \\ 1\cos(t) + 1\sin(t), t \in \left[\pi, \frac{3\pi}{2}\right] \\ 2\cos(t) + 1\sin(t), t \in \left[\frac{3\pi}{2}, 2\pi\right] \end{cases} \tag{11}$$

The interval solution $\mathbf{y}(t) = [\underline{y}(t), \bar{y}(t)]$ is shown on the Fig. 1. It is interesting to note that the interval solution is continuous but non-differentiable. Interval solution can be created by using 4 different point solutions (compare Fig. 2). Because the function $y = y(t,p)$ is monotone (as a function of variable p), the interval solution is exact. It is possible to verify the interval solution by using more simulations (compare Fig. 3). For each timestep t it is possible to find such combination of parameters $p^{min}(t), p^{max}(t)$ such that (compare equation (8))

$$\underline{y}(t) = y(t, p^{min}(t)),\ \overline{y}(t) = y(t, p^{max}(t)) \tag{12}$$

Specific form of the formulas (12) is shown in the equations (10, 11). Because the solution is monotone (with respect to the variables p, then the functions $p^{min}(t)$,$p^{max}(t)$ have finite number of values. If we know these functions we can calculate the interval solution (if appropriate method of solution exist).

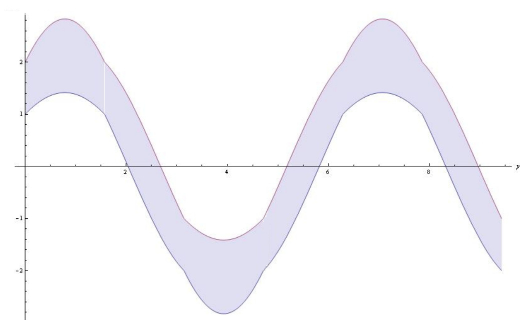

Fig. 1. Interval solution of the equation 9

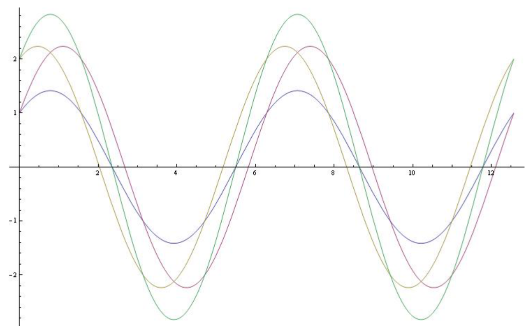

Fig. 2. Interval solution of the equation 9 can be calculated by using 4 solutions

B. Damped vibrations

Let us consider the following differential equation

$$\frac{dy^2}{dt^2} + 2\beta\frac{dy}{dt} + \omega_0^2 y = 0 \tag{13}$$

General solution of that equation can be written in the following form

$$y = Ae^{-\beta t}\sin\left(\omega t + \varphi\right) \tag{14}$$

In calculations we assume that $A \in \left[\underline{A}, \overline{A}\right] = [1, 2]$, $\beta \in [0.5, 0.6]$, $\varphi = 0$, $\omega = 1$. In this case the solution is also monotone with respect to the interval parameters. Because of that, it is possible to calculate the exact solution in the following way

$$\underline{y}(t) = \begin{cases} 1e^{-0.5t}\sin\left(t\right), t \in \left[0, \frac{\pi}{2}\right] \\ 1e^{-0.6t}\sin\left(t\right), t \in \left[\frac{\pi}{2}, \pi\right] \\ 2e^{-0.6t}\sin\left(t\right), t \in \left[\pi, \frac{3\pi}{2}\right] \\ 2e^{-0.5t}\sin\left(t\right), t \in \left[\frac{3\pi}{2}, 2\pi\right] \end{cases} \tag{15}$$

$$\bar{y}(t) = \begin{cases} 2e^{-0.6t}\sin\left(t\right), t \in \left[0, \frac{\pi}{2}\right] \\ 2e^{-0.5t}\sin\left(t\right), t \in \left[\frac{\pi}{2}, \pi\right] \\ 1e^{-0.5t}\sin\left(t\right), t \in \left[\pi, \frac{3\pi}{2}\right] \\ 1e^{-0.6t}\sin\left(t\right), t \in \left[\frac{3\pi}{2}, 2\pi\right] \end{cases} \tag{16}$$

The interval solution $\mathbf{y}(t) = [\underline{y}(t), \bar{y}(t)]$ is shown on the Fig. 4. In order to get the exact interval solution it is enough to find only 2 solutions (compare Fig. 5).

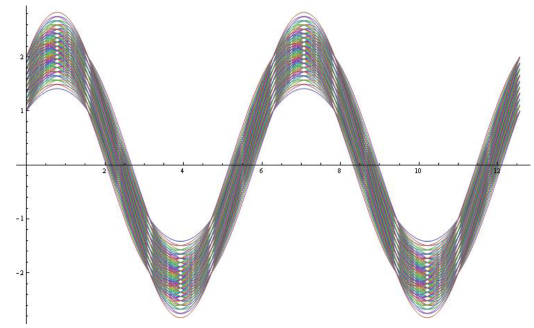

Fig. 3. Interval solution of the equation 9. 10 solutions which can be used as veryfication of our results.

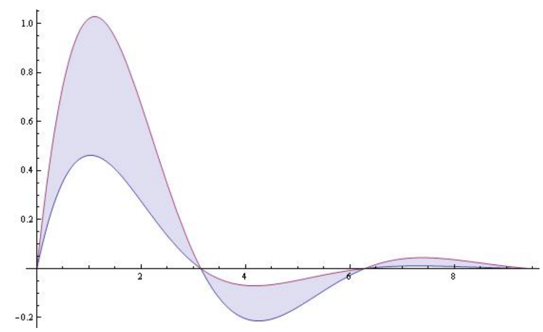

Fig. 4. Interval solution of the equation 13.

C. Free vibrations - non-monotone solution

Let us consider the equation (9) with the following initial conditions $y(0) = 0$, $\frac{dy(0)}{dt} = \omega = p \in [0.5, 0.6]$. The solution is shown on the Fig. 6. In this case the sign of partial derivative $\frac{\partial y}{\partial p}$ is not constant. Because of that, it is not possible to get the interval solution by using finite number of point solutions (compare Fig. 6). If we try to calculate the solution by using equations $\underline{y}(t) = y(t, p^{min}(t))$, $\overline{y}(t) = y(t, p^{max}(t))$, then the functions $p^{min}(t), p^{max}(t)$ have infinitely many values, which is a big problem in calculations and future applications. In other words, it is not possible to get the exact interval solution $[\underline{y}(t), \overline{y}(t)]$ by using finite number of point solutions (compare Fig. 6).

In application it is important to find the interval solution $[\underline{y}(t), \overline{y}(t)]$ as well as appropriate combinations of parameters $p^{min}(t), p^{max}(t)$, which generate these solutions.

III. Numerical method

In previous section we discussed analytical methods for solution of differential equations. Unfortunately in applications usually it is not possible to get the analytical solution. In that situations we have to apply the numerical methods. Let us consider a system of first order parameter dependent differential equations

$$\begin{cases} \frac{dy}{dt} = f(t, y, p) \\ x(0, p) = x_0(p) \\ p \in \mathbf{p} \end{cases} \tag{17}$$

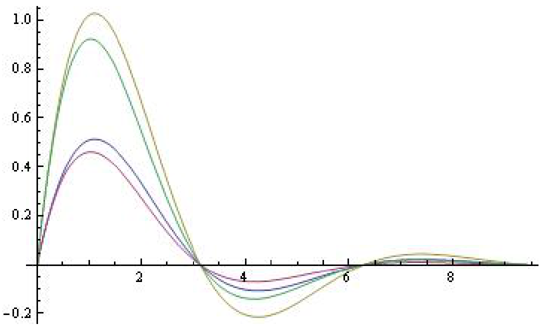

Fig. 5. Interval solution of the equation 13 can be calculated exactly by using point solutions.

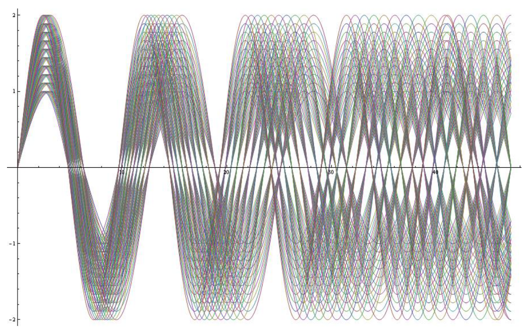

Fig. 6. Interval solution of the equation 13 can be calculated exactly by using point solutions.

Partial derivative with respect to the uncertain parameter $v_i = \frac{\partial y}{\partial p_i}$ satisfy the following differential equation.

$$\begin{cases} \frac{dv_i}{dt} = \frac{\partial f(t,y,p)}{\partial p} + \frac{\partial f(t,y,p)}{\partial y} v_i \\ v_i(0,p) = v_{0i}(p) = \frac{\partial x_0(p)}{\partial p_i} \\ p \in \mathbf{p} \end{cases} \tag{18}$$

If the function $y = y(t,p)$ is monotone (with respect to the variable p), then in order to find the interval solution it is necessary to find the solution of the equation (18) for the mid point $p_0 = mid(\mathbf{p})$ (i.e. $y = y(t,p_0)$ and $v_i = v_i(t,p_0)$. Then it is necessary to find all combinations of parameters, which correspond to the sign of the function of $v_i = v_i(t,p_0) = \frac{\partial y(t,p_0)}{\partial p_i}$.

$$p(t)^{min} = \underline{p}_i, p(t)^{max} = \overline{p}_i, \quad if \ \frac{\partial y(t,p_0)}{\partial p_i} \geqslant 0 \tag{19}$$

$$p(t)^{min} = \overline{p}_i, p(t)^{max} = \underline{p}_i, \quad if \ \frac{\partial y(t,p_0)}{\partial p_i} < 0 \tag{20}$$

If the functions $p(t)^{min}, p(t)^{max}$ have finite number of values, then it is possible to find the exact interval solution by using appropriate endpoints $\underline{y}(t) = y(t,p^{min}(t))$, $\overline{y}(t) = y(t,p^{max}(t))$. Solutions $y(t,p^{min}(t)), y(t,p^{max}(t))$ can be calculated by using any numerical method.
Unfortunately usually the function $y = y(t,p)$ is not monotone (with respect to the parameter p). Methods which can be applied in these situations will be presented in the next sections.

IV. Hermite interpolation method

Let us consider the following differential equation

$$\begin{cases} \frac{dy}{dt} = p \cdot cos(pt) \\ y(0) = 0 \\ p \in [\underline{p}, \overline{p}] \end{cases} \tag{21}$$

Exact solution is equal to $y = sin(pt)$ and we can use it in verification of the results. Solution is not monotone with respect to the parameter p. In this approach we are going to solve the equation (21) for $p = mid(\mathbf{p})$. After that it is necessary to find partial derivative $\frac{\partial y}{\partial p}$, which satisfies the following differential equation

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial y}{\partial p}\right) = cos(pt) - ptsin(pt) \\ \frac{\partial y(0)}{\partial p} = 0 \\ p \in [\underline{p}, \overline{p}] \end{cases} \tag{22}$$

Let's find the solution of the equation (21) and equation (22) for the mid point $p^{(0)} = mid(\mathbf{p})$. Using that solution and appropriate derivative it is possible to create interpolation polynomial.

$$y(t,p) \approx y^{approx}(t,p^{(0)},p) \tag{23}$$

In order to increase the accuracy of the calculations it is possible to use more solutions, which correspond to different values $p^{(0)}, ..., p^{(n)}$.

$$y(t,p) \approx y^{approx}(t,p^{(0)},...,p^{(n)},p) \tag{24}$$

Let's introduce the following lists

$$L_p = \{p^{(0)}, ..., p^{(n)}\} \tag{25}$$

$$L_y = \{y(t,p^{(0)}), y(t,p^{(1)}), ..., y(t,p^{(n)})\} \tag{26}$$

$$L_{\frac{\partial y}{\partial p_i}} = \left\{ \frac{\partial y(t,p^{(0)})}{\partial p_i}, \frac{\partial y(t,p^{(1)})}{\partial p_i}, ..., \frac{\partial y(t,p^{(n)})}{\partial p_i} \right\} \tag{27}$$

In this approach it is necessary to assume some initial form of the approximate solution

$$y^{approx}(t,p) = \Phi_t(a_1, ..., a_m, p) \tag{28}$$

unknown parameters $a_1, ..., a_m$ can be calculated from the following interpolation conditions

$$y(t,p_i) = y^{approx}(t,p^{(0)},...,p^{(n)},p_i), \tag{29}$$

$$\frac{\partial y(t,p_i)}{\partial p} = \frac{\partial y^{approx}(t,p^{(0)},...,p^{(n)},p_i)}{\partial p} \tag{30}$$

Conditions (29,30) have to be applied for each timestep separately. It is also possible to apply interpolation for the variables t and p simultaneously. Now using classical optimization methods (in this paper the gradient descent method was applied) it is possible to find approximate values of upper and lower bound of the solution

$$\underline{y}^{approx} = min\{y^{approx}(t,p^{(0)},...,p^{(n)},p) : p \in \mathbf{p}\}, \tag{31}$$

$$\overline{y}^{approx} = max\{y^{approx}(t,p^{(0)},...,p^{(n)},p) : p \in \mathbf{p}\}. \tag{32}$$

$$\underline{p}_{approx}^{min}(t) = argmin\{y^{approx}(t,p) : p \in \mathbf{p}\}, \quad (33)$$

$$\overline{p}_{approx}^{max}(t) = argmax\{y^{approx}(t,p) : p \in \mathbf{p}\}. \quad (34)$$

For $p \in [1, 1.2]$ example solution is shown on the Fig. 7. For wider intervals the solution contain all values between

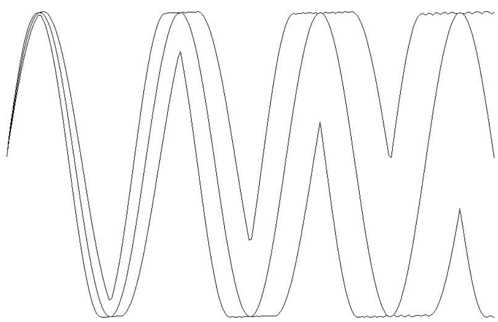

Fig. 7. Interval solution of the equation 21 for $p \in [1, 1.2]$.

$[-1, 1]$ (compare Fig. 8). It is interesting to note that even for

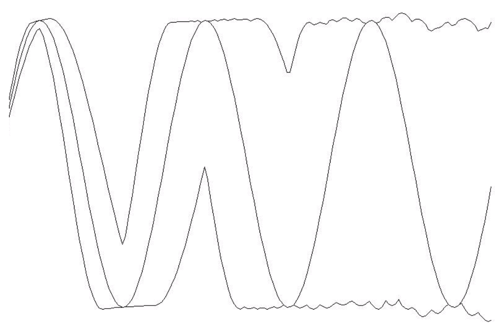

Fig. 8. Interval solution of the equation (21) for $p \in [1, 1.5]$.

narrow interval data after some time the interval solution of the equation (21) is $\underline{y} = -1, \overline{y} = 1$. However, it is very hard to use that kind of solution in the practical applications.

In presented calculations three solutions and their derivatives were applied $y(t,p^{(0)}),\frac{\partial y(t,p^{(0)})}{\partial p}$,$y(t,p^{(1)}),\frac{\partial y(t,p^{(1)})}{\partial p}$, $y(t,p^{(2)})$, $\frac{\partial y(t,p^{(2)})}{\partial p}$ were $p^{(0)} = mid(\mathbf{p})$, $p^{(1)} = \underline{p}$, $p^{(2)} = \overline{p}$. Using point solutions it is possible to calculate reliable inner estimation of the interval solution

$$\underline{y}^{inner}(t, p^{(0)}, ..., p^{(n)}) = min\{y(t, p^{(i)}) : i = 1, ..., n\} \quad (35)$$

$$\overline{y}^{inner}(t, p^{(0)}, ..., p^{(n)}) = max\{y(t, p^{(i)}) : i = 1, ..., n\} \quad (36)$$

Let's calculate the combination of parameters $p_{new}^{min}, p_{new}^{max}$ which correspond to the maximum difference between the interval solution $(31, 32)$ and the inner solution

$$t_{min} = \underset{t\in[t_1,t_2]}{argmax}\{\underline{y}^{approx}(t) - \underline{y}^{inner}(t)\}, \quad (37)$$

$$\underline{y}^{approx}(t_{min}) = y(t, p_{new}^{min}), \quad (38)$$

$$t_{max} = \underset{t\in[t_1,t_2]}{argmax}\{\overline{y}^{approx}(t) - \overline{y}^{inner}(t)\}, \quad (39)$$

$$\overline{y}^{approx}(t_{max}) = y(t, p_{new}^{max}). \quad (40)$$

Now, it is necessary to calculate appropriate values of the function y (i.e. $y(t,p_{new}^{min}), y(t,p_{new}^{max})$) and derivatives (i.e. $\frac{\partial y(t,p_{new}^{min})}{\partial p}$, $\frac{\partial y(t,p_{new}^{min})}{\partial p}$), then add them to the lists $L_p, L_y, L_{\frac{\partial y}{\partial p}}$. If the difference $\underline{y}^{approx}(t) - \underline{y}^{inner}(t)$ and $\overline{y}^{approx}(t) - \overline{y}^{inner}(t)$ is sufficiently small, then we can stop calculations and assume that

$$\underline{y}(t) \approx \underline{y}^{approx}, \overline{y}(t) \approx \overline{y}^{approx}(t) \quad (41)$$

V. Adaptive Taylor method

The algorithm which we described in the last section is very accurate even for large uncertainty, unfortunately interpolation (29,30) is very complicated when the amount of uncertain parameters is large. In order to simplify the method for calculation of $y^{approx}(t)$ in this section Taylor series will be applied.

A. First order approximation

In the simplest case in order to create $y^{approx}(t)$ first order Taylor series can be applied. For all points in the list $L_p = \{p^{(0)}, ..., p^{(n)}\}$ it is necessary to calculate appropriate values of the lists L_y and $L_{\frac{\partial y}{\partial p_i}}$. Let's create a new list L_{T_1} which contain first order Taylor approximation of the solution $y(t, p^{(k)})$

$$T_1(t, p^{(k)}, p) = y(t, p^{(k)}) + \sum_i \frac{\partial y(t, p^{(k)})}{\partial p_i}(p_i - p_i^{(k)}) \quad (42)$$

$$L_{T_1} = \{T_1(t, p^{(0)}, p), ..., T_1(t, p^{(n)}, p)\} \quad (43)$$

Accuracy of the Taylor expansion depend on the distance $\rho(p, p^{(k)}) = \|p - p^{(k)}\|$. Because of that in order to estimate the value of the function $y^{approx}(t, p)$, it is good to apply the closest Taylor expansion to the point p. So, in order to calculate the value $y^{approx}(t, p)$ it is necessary to calculate

$$k_{min} = \underset{k\in 0,...,n}{argmin}\|p - p^{(k)}\| \quad (44)$$

and then

$$y^{approx}(p) = y^{approx}(t, p^{(k_{min})}, p) \quad (45)$$

It is also possible to calculate the value of the approximate solution by using weighted average

$$y^{approx}(t, p) = \frac{\sum_i T_1(t, p^{(i)}, p)w_i(p, p^{(i)})}{\sum_i w_i(p, p^{(i)})} \quad (46)$$

where

$$y^{approx}(t, p) = \frac{1}{\|p - p^{(i)}\|^{\alpha}} \quad (47)$$

Usually $\alpha = 2$. Computational algorithm which produce upper and lower bound is the same like in the section IV.

B. Second and higher order approximation

In second order approximation it is necessary to use list of second order derivatives $L_{\frac{\partial^2 y}{\partial p_i \partial p_j}}$ and also second order Taylor expansions L_{T_2}. It is also possible to use higher order Taylor expansions in a similar way.

C. Numerical example - forced vibrations

Let us consider forced vibrations

$$\frac{d^2y}{dt^2} + \frac{k}{m}y = \frac{P}{m}\cos(\omega t) \tag{48}$$

where $y(0) = 1, \frac{dy(0)}{dt} = 0$, P=1, $k \in [1, 2]$, $m \in [1, 2]$, $\omega = 2$. Solutions $\underline{y}^{approx}$ and $\overline{y}^{approx}$, which were calculated by using only the mid point, are shown on the Fig. 9. In order to increase accuracy it is possible to calculate upper and lower bound $\underline{y}^{approx}$ and $\overline{y}^{approx}$ using most promising combinations of parameters $(p_{new}^{min}, p_{new}^{max})$. Then, it is possible to calculate the interval solution once again. The results of the calculations (for 3 point solutions) is shown on the Fig. 10. In order to increase accuracy it is possible to calculate new combinations of parameters $(p_{new}^{min}, p_{new}^{max})$, which reduce the error. Interval solutions which use 5 point solutions is shown on the Fig. 11.

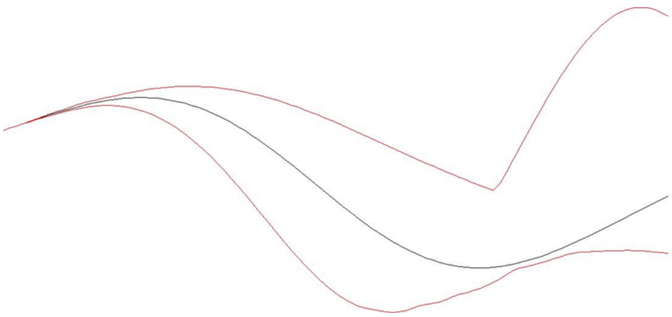

Fig. 9. Interval solution of the equation (48) which use only mid point solution and second order approximation.

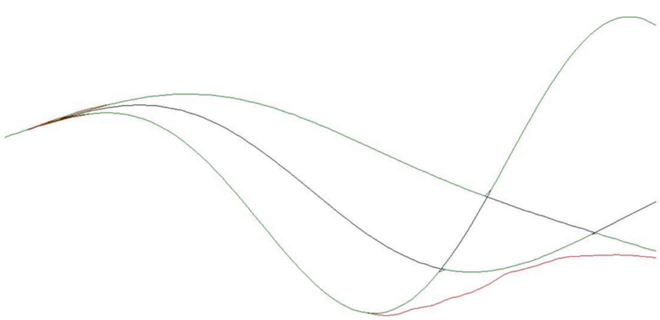

Fig. 10. Interval solution of the equation (48) which use three point solutions and second order approximation.

Lets define the maximum error as a maximum difference between the approximate solution y^{approx} and the inner solution y^{inner}. Error has to be calculated separately for the upper bound and lower bound.

$$e^{min} = \max_t |\underline{y}^{approx}(t) - \underline{y}^{inner}(t)| \tag{49}$$

$$e^{max} = \max_t |\overline{y}^{approx}(t) - \overline{y}^{inner}(t)| \tag{50}$$

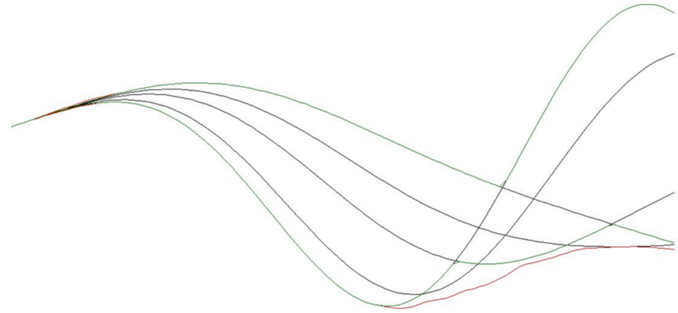

Fig. 11. Interval solution of the equation (48) which use five point solutions and second order approximation.

TABLE I
ESTIMATED ERROR OF THE SOLUTION

Number of solutions	e^{min}	e^{max}
3	4.64192	1.96693
5	0.04037	0.71053

Values of that error for differen number of solutions is shown in the table below.

The difference between the true solution and the approximate solution ($\underline{y} - \underline{y}^{approx}$, $\overline{y} - \overline{y}^{approx}$) is much smaller. On the foregoing pictures we see convergence of the method.

D. Numerical example - vibrations of beam

Let us consider a beam which is shown on the Fig. 12. Partial differential equation of vibrations can be written in the following form

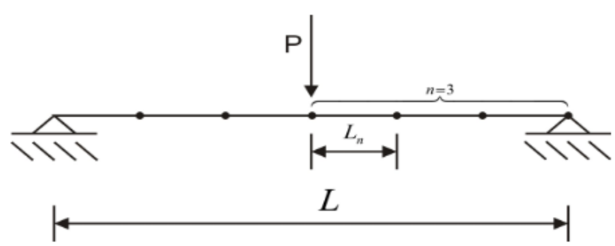

Fig. 12. Beam FEM model.

$$\begin{cases} -EJ\frac{\partial^4 y}{\partial x^4} + q = \rho A\frac{\partial^2 y}{\partial t^2} \\ y(0,t) = 0, y(L,0) = 0 \\ \frac{\partial^2 y(0,t)}{\partial x^2} = 0, \frac{\partial^2 y(L,t)}{\partial x^2} = 0 \\ y(x,0) = 0 \\ q(x,t) = \delta(x - L/2)P(H(t-t_0) - H(t)) \end{cases} \tag{51}$$

After FEM discretization [6] of the equation (51) we get the following ordinary differential equation

$$M\ddot{y} + Ky = Q \tag{52}$$

where M is the mass matrix, K is he stiffness matrix, and Q is the vector of forces. Now using numerical integration it is possible to calculate the solution. In this example the

following interval data (5% uncertainty) were applied $E \in [190 \cdot 10^9, 210 \cdot 10^9][Pa]$, $J \in [7.92 \cdot 10^{-6}, 8.75 \cdot 10^{-6}][m^4]$, $A \in [0.0095, 0.0105][m^2]$, $\rho \in [7480.3, 8267.7]\left[\frac{kg}{m^3}\right]$. Other parameters are the following $L = 10[m]$, $P = 1000[N]$, $dt = 0.001[s]$, $t_0 = 0.001[s]$.
In the first part of the algorithm it is necessary to find partial derivatives of the solution with respect to the uncertain parameters p_i. In order to do that it is possible to use finite difference.

$$\frac{\partial y(t,x,p_0)}{\partial p_i} \approx \frac{y(t,x,...,p_i^{(0)}+\Delta p_i^{(0)},...)-y(t,x,...,p_i^{(0)},...)}{\Delta p_i^{(0)}} \tag{53}$$

All solutions $y(t,x,...,p_i^{(0)}+\Delta p_i^{(0)},...)$ and $y(t,x,...,p_i^{(0)},...)$ should be added to the list L_y as well as all points p_0, $p_i^{(0)} + \Delta p_i^{(0)}$ should be added to the list L_p. Interval solution is shown in the Fig. 13. On the picture it is possible to see the mid point

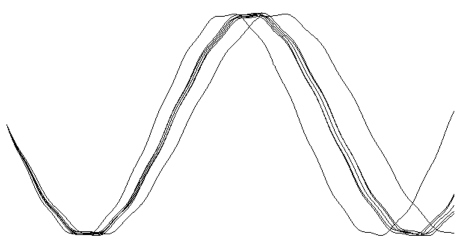

Fig. 13. Interval solution of the equation (51).

solution as well as all solutions $y(t,x,...,p_i^{(0)} + \Delta p_i^{(0)},...)$ which are necessary to find the derivatives (53). In this case in order to find $y^{approx}(t,x,p)$ first order Taylor approximation was applied.

VI. Web Application

Examples which are presented in this paper are available on-line on the following web page. http://andrzej.pownuk.com/interval_web_applications.htm.
Web applications were implemented by using asp.net and Silverlight. Example web application is shown on Fig. 14.

Conclusions

In this paper a new approximate method for calculating solution of the interval differential equations $[\underline{y}^{approx}, \overline{y}^{approx}]$ was presented. The method allows as to calculate guaranteed inner bound of the solution $[\underline{y}^{inner}, \overline{y}^{inner}]$. Using the differences $\underline{y}^{approx} - \underline{y}^{inner}$, $\overline{y}^{approx} - \overline{y}^{inner}$ it is possible to estimate accuracy of the calculations and increase accuracy in the next iteration.
According to the numerical results the interval solution is very often a non-differentiable function. If the solution is monotone, then the results of the calculations are exact.
The method allows us to calculate not only the interval solution $[\underline{y}^{approx}, \overline{y}^{approx}]$ but also combinations of parameters $p^{min}_{approx}(t), p^{max}_{approx}(t)$, which generate each bound.

$$\underline{y}^{approx} = y(t, p^{min}_{approx}(t)), \overline{y}^{approx} = y(t, p^{max}_{approx}(t)) \tag{54}$$

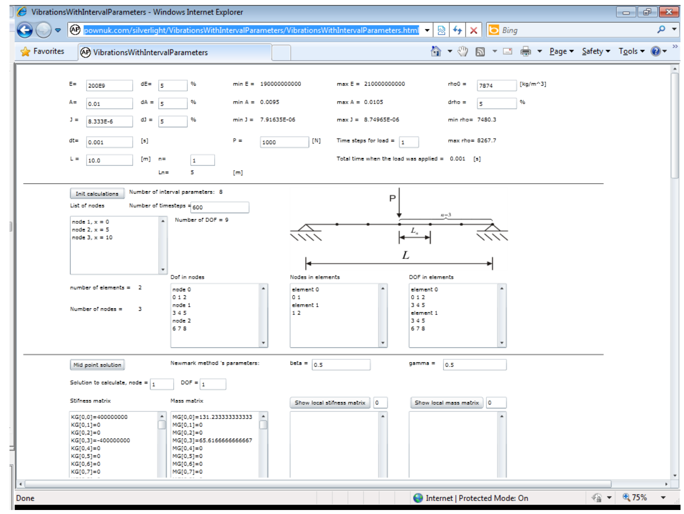

Fig. 14. Web application for modeling of the vibrations of beams with the interval parameters (Silverlight).

$$\underline{y}^{inner} = y(t, p^{min}_{inner}(t)), \overline{y}^{inner} = y(t, p^{max}_{inner}(t)) \tag{55}$$

This is very important in the applications of the interval methods.
Presented method can be applied in the case of very large uncertainty. Adaptive approximation (the method which was described in this paper) can be applied to the solution of ordinary as well as partial differential equations with the interval parameters.

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