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RECENT ADVANCES IN APPLIED MATHEMATICS

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Finite Element Method with the Interval Set Par... and its Applications in Computational Scie...

ANDRZEJ POWNUK
The University of Texas at El Paso
Department of Mathematical Sciences
500 West University Avenue, El Paso, Texas
USA
andrzej@pownuk.com, http://andrzej.pownuk.com

BEHZAD DJAFARI-RO...
The University of Texas ...
Department of Mathematica...
500 West University Avenue, ...
USA
behzad@math.utep.edu

NAVEEN KUMAR GOUD RAMUNIGARI
The University of Texas at El Paso
Department of Civil Engineering
500 West University Avenue, El Paso, Texas
USA
r.naveengoud@gmail.com

Abstract: The Finite Element Method (FEM) is one of the most popular approach to describe ... today. In order to apply this method efficiently, it is necessary to know the exact values ... case of uncertain shapes, the FEM method leads to a parameter dependent system of alg... interval set parameters. In this paper the solutions for such equation will be presented. The ... of topological derivative and monotonicity. Numerical examples will be presented.

Key–Words: Interval sets, uncertainty, interval functional parameters, finite element method

1 Engineering problems with the uncertain shape

Almost all engineering problems require a very precise information about the geometry (eg. height, thickness, curvature, coordinate of the characteristic points of the structure etc.) of the problem. Unfortunately, due to many reasons (unavoidable inaccuracy in the construction process, bad materials, etc.) the real dimension of the engineering structure are not know exactly [5, 6, 12].

Civil engineering projects are usually very unique. Because of that it is very hard to get reliable probabilistic characteristics of the structures. One of the simplest methods for modeling uncertainty is based on the intervals. If Ω denotes the domain of the structure, then, due to uncertainty, we can assume that

$$\Omega \in [\underline{\Omega}, \overline{\Omega}] \tag{1}$$

where $\underline{\Omega}, \overline{\Omega}$ denotes the extreme value of the shapes. If $u = u(x, \Omega)$ is a characteristic of the structure, e.g. displacement, then in the case of the uncertainty, instead of one number we have the whole interval

$$[\underline{u}(x), \overline{u}(x)] = \{u(x, \Omega) : \Omega \in [\underline{\Omega}, \overline{\Omega}]\} \tag{2}$$

In this paper, some procedu... $[\underline{u}(x), \overline{u}(x)]$ will be presented. ... tain parameters were consider...

2 Examples of set ... tions

2.1 Center of gravity

x coordinate of the center of ... of a set dependent function

$$x_C(\Omega) = \frac{\int_\Omega x d\mu}{\int_\Omega d\mu}$$

2.2 Moment of inertia

Different kinds of moment of ... by using integrals, and they are ...

$$I_y(\Omega) = \int_\Omega x^2 d\mu, \quad I_0(\Omega) = \ldots$$

3.2 Solution of PDE or integral equations

Boundary value problems (BVP) can be described as system of partial differential equations

$$\begin{cases} A(x)u(x) = f_a(x), & for\ x \in \Omega \\ B(x)u(x) = f_b(x), & for\ x \in \partial\Omega \end{cases} \tag{5}$$

where $A(x)u(x) = f_a(x)$ is a PDE, which is defined in the domain $Int(\Omega)$, and $B(x)u(x) = f_b(x)$ is a BC which is defined on the boundary $\partial\Omega$. The Solution of BVP is a set dependent function $u = u(x, \Omega)$

For example, the solution of plate equation

$$\begin{cases} \Delta^2 w(x) = \frac{q(x)}{D}, & for\ x \in \Omega \\ w(x) = w^*(x), & for\ x \in \partial\Omega \end{cases} \tag{6}$$

is a set dependent function $w = w(x, \Omega)$ (w is a small transverse, out-of-plane displacement, p is a distributed load, D is the bending/flexural rigidity, $w^*(x)$ is a known displacement at the boundary of the plate

3 Classical definition of topological derivative

Let us consider an open, bounded domain, $\Omega \subset R^n$ with a smooth boundary $\partial\Omega$. If the domain is perturbed by introducing a small hole B_ε of radius ε at an arbitrary point $x \in \Omega$, we have new domain $\Omega_\varepsilon = \Omega - \bar{B}_\varepsilon$, whose boundary is denoted by $\Gamma_\varepsilon = \partial\Omega \cup \partial B_\varepsilon$. Topological derivative of certain set function $\psi = \psi(\Omega)$ can be defined as the following [3, 8]

$$D_T(x) = \lim_{\varepsilon \to 0^+} \frac{\psi(\Omega_\varepsilon) - \psi(\Omega)}{f(\varepsilon)} \tag{7}$$

where $f(\varepsilon)$ is a given function, which is positive and

$$\lim_{\varepsilon \to 0^+} f(\varepsilon) = 0. \tag{8}$$

If $\Omega_\varepsilon = \Omega - \bar{B}_\varepsilon$ then we can use the following notation

$$D_T(x) = \frac{d\psi}{d\Omega(x)} \tag{9}$$

4 Parametric method for calculating topological derivative

It is possible to define topological derivative for completely arbitrary perturbations. In this case, Ω_ε is the arbitrary set (i.e. not necessarily $\Omega_\varepsilon = \Omega - \bar{B}_\varepsilon$). However $\Omega_\theta \to \Omega$, when $\theta \to 0$.

$$D_T(x) = \lim_{\theta \to 0} \frac{\psi(\Omega_\theta) - \psi(\Omega)}{f(\theta)} = \tag{10}$$

$$= \lim_{\theta \to 0} \frac{\frac{\psi(\Omega_\theta) - \psi(\Omega)}{\theta}}{\frac{f(\varepsilon) - f(0)}{\theta}} = \left(\frac{\frac{d\psi}{d\theta}}{\frac{df}{d\theta}} \right)_{\theta=0} = D_T^{(\theta)}(x) \tag{11}$$

In some cases, the formula (11) gives the same results for different parameterizations (ε).

Let us consider a triangle ABC, where A=(0,0), B=(1,0), C=(1+ε,1). and a function $\psi_1(\varepsilon) = \psi_1(\Omega_\varepsilon) = |\Omega_\varepsilon|^2$, where $|\Omega_\varepsilon| = (1+\varepsilon)/2$ is the area of the triangle, $f(\varepsilon) = |\Omega_\varepsilon| - 0.5$.

$$D_T^\varepsilon = \left(\frac{\frac{d\psi_1}{d\varepsilon}}{\frac{df}{d\varepsilon}} \right)_{\varepsilon=0} = \left(\frac{\frac{1+\varepsilon}{2}}{\frac{1}{2}} \right)_{\varepsilon=0} = 1.0 \tag{12}$$

In this case, topological derivative can be calculated for all parameterisations

$$\lim_{\varepsilon \to 0^+} \frac{\psi_1(\Omega_\varepsilon) - \psi_1(\Omega)}{f(\varepsilon)} = \lim_{\varepsilon \to 0^+} \frac{|\Omega_\varepsilon|^2 - 0.5^2}{|\Omega_\varepsilon| - 0.5} = \tag{13}$$

$$\lim_{\varepsilon \to 0^+} |\Omega_\varepsilon| + 0.5 = |\Omega_0| + 0.5 = 1 \tag{14}$$

Let us consider the function $\psi_2(\Omega_\varepsilon) = y_C = \varepsilon$, for the parameterisation which was given above

$$D_T^\varepsilon = \left(\frac{\frac{d\psi_2}{d\varepsilon}}{\frac{df}{d\varepsilon}} \right)_{\varepsilon=0} = \left(\frac{1}{\frac{1}{2}} \right)_{\varepsilon=0} = 2 \tag{15}$$

Let us consider different parameterisation of the shape of the triangle C=(1 + γ,1). In this case, $\psi_2(\Omega_\gamma) = y_C = 1$

$$D_T^\gamma = \left(\frac{\frac{d\psi_2}{d\gamma}}{\frac{df}{d\gamma}} \right)_{\gamma=0} = \left(\frac{0}{\frac{1}{2}} \right)_{\varepsilon=0} = 0 \tag{16}$$

Then $2 = D_T^\varepsilon \neq D_T^\gamma = 0$ i.e. the result depends on the parameterisation.

In the literature, usually the concept of parameter independent topological derivative is used [3].

5 Basic formulas for calculating topological derivatives

5.1 Function in the form $\psi(\Omega) = \int\limits_\Omega L(x)dx$

Theorem 1 *Let us consider the integral in the form*

$$\psi(\Omega_\varepsilon) = \int\limits_{\Omega_\varepsilon} L(x)dx \tag{17}$$

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where L is a continuous function and $\Omega \subset R^m$ is a sufficiently regular set and Ω_ε=$\Omega - B_\varepsilon$. The topological derivative of the function ψ in the point $y \in \Omega$ is equal to [4]

$$\frac{d\psi(\Omega)}{d\Omega(y)} = \frac{d}{d\Omega(y)} \int\limits_{\Omega} L(x)dx = L(y) \tag{18}$$

Example

$$\frac{d}{d\Omega(x_1, x_2)} \int\limits_{\Omega} (x_1^2 + x_2^2)dx = x_1^2 + x_2^2 \tag{19}$$

5.2 Functions of in the form $\psi(\Omega) = F\left(\int\limits_{\Omega} L(x)dx\right)$

Theorem 2 *Let us consider the integral in the form*

$$\psi(\Omega) = F\left(\int\limits_{\Omega} L(x)dx\right) \tag{20}$$

where $L : R^m \to R$ is a continuous function, $F : R \to R$ is differentiable function and $\Omega \subset R^m$ is a sufficiently regular set and Ω_ε=$\Omega - B_\varepsilon$. The topological derivative of the function ψ in the point $y \in \Omega$ is equal to

$$\frac{d\psi(\Omega)}{d\Omega(y)} = F'\left(\int\limits_{\Omega} L(x)dx\right) \cdot L(y) \tag{21}$$

This is a consequence of the chain rule.
Example

$$\frac{d}{d\Omega(y)}\left(\int\limits_{\Omega} x_i^2 dx\right)^3 = 3\left(\int\limits_{\Omega} x_i^2 dx\right)^2 \cdot y_i^2 \tag{22}$$

It is important to distinguish between general parameterization Ω_θ and Ω_ε=$\Omega - B_\varepsilon$.

Theorem 3 *Let us consider the integral in the form*

$$\psi(\Omega_\theta) = \int\limits_{\Omega_\theta} L(x, \theta)dx \tag{23}$$

where L is a continuous function, and $\Omega \subset R^m$ is a sufficiently regular set. For the general parameterization θ topological derivative of the function ψ

$$\frac{d}{d\theta}\int\limits_{\Omega_\theta} Ldx = \int\limits_{\Omega} \frac{\partial L}{\partial \theta}dx + \int\limits_{\partial\Omega} Lvnds \tag{24}$$

where $v = \frac{\partial r}{\partial \theta}$ and $r = r(x, \theta)$ is the [illegible] *scription of the boundary $\partial\Omega$ ($x \in \partial\Omega$)* [illegible] *normal vector to the boundary.*

This is Reynolds transport theorem [3, 8].

Theorem 4 *Let*

$$\psi(\Omega_\varepsilon) = \int\limits_{\Omega_\varepsilon} L(x, \varepsilon)dx$$

and Ω_ε=$\Omega - B_\varepsilon$ then

$$\frac{d\psi}{d\Omega(y)} = \int\limits_{\Omega} \frac{\frac{\partial L(x,0)}{\partial \varepsilon}}{\frac{\partial |\Omega_\varepsilon|}{\partial \varepsilon}}dx + L(y,0)$$

Theorem 4. can be extended to the [illegible] derivative.

Theorem 5 *Let*

$$\psi = \psi(\Omega_\varepsilon) = \int\limits_{\Omega_\varepsilon} L(x, \Omega_\varepsilon)dx$$

and Ω_ε=$\Omega - B_\varepsilon$ then

$$\frac{d\psi}{d\Omega(y)} = \int\limits_{\Omega} \frac{\frac{\partial L(x,0)}{\partial \varepsilon}}{\frac{\partial |\Omega_\varepsilon|}{\partial \varepsilon}}dx + L(y, \Omega \ldots$$

6 Center of gravity

6.1 Topological derivative

The center of gravity (i-th coordinate) of the [illegible] be calculated in the following way

$$x_i^C(\Omega) = \frac{\int\limits_{\Omega} x_i dx}{\int\limits_{\Omega} dx}$$

The topological derivative can be calculated [illegible] ing quotient rule

$$\frac{dx_i^C}{d\Omega(x)} = \frac{x_i \int\limits_{\Omega} dx - \int\limits_{\Omega} x_i dx}{\left(\int\limits_{\Omega} dx\right)^2}$$

The sign of the topological derivative [illegible] same like the sign of the difference

$$x_i \int\limits_{\Omega} dx - \int\limits_{\Omega} x_i dx.$$

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7 The implicit topological derivative

The finite element method lead to the following parameter dependent system of equations [10].

$$K(\Omega)u = Q(\Omega) \tag{32}$$

where $K(\Omega)$ is the local stiffness matrix, $Q(\Omega)$ is the load vector, u is the displacement vector. using, the implicit function theorem, it is possible to calculate topological derivative of the vector u.

$$K(\Omega)\frac{du}{d\Omega(x)} = \frac{dQ(\Omega)}{d\Omega(x)} - \frac{dK(\Omega)}{d\Omega(x)} \cdot u(\Omega) \tag{33}$$

8 Interval displacements in 2D elasticity; problem with uncertain shape

Let us consider the rectangular FEM element [10]. The stiffness matrix can be defined in the following way

$$K = \int_{\Omega} B^T D B dV \tag{34}$$

Derivative of the integral (34) can be calculated by using Reynold's Transport Theorem.

$$\frac{dK}{d\theta} = \int_{\Omega} \frac{d}{d\theta}(B^T D B) dV + \int_{\partial\Omega} B^T D B v n dV \tag{35}$$

Derivative $\frac{dK}{d\theta}$ can be also calculated directly, if the analytical expression for K is known. It is possible to apply the numerical differentiation e.g.

$$\frac{dK}{d\theta} \approx \frac{K(\theta + \Delta\theta) - K(\theta)}{\Delta\theta} \tag{36}$$

Let us assume that θ is a y coordinate of the node 3. Matrix D is constant, then $\frac{dD}{d\theta} = 0$. Matrix B is defined by using derivatives of the shape functions $\frac{\partial N_i}{\partial x}$. For rectangular element the first shape function has the following form

$$N_1 = \left(1 - \frac{x - x_1}{x_2 - x_1}\right)\left(1 - \frac{y - y_1}{y_2 - y_1}\right) \tag{37}$$

$$N_1 = \left(1 - \frac{x - x_1}{x_2 - x_1}\right)\left(1 - \frac{y - y_1}{y_2 + \theta - y_1}\right) \tag{38}$$

$$|\Omega_\theta| = (x_2 - x_1)(y_2 + \theta - y_1) \tag{39}$$

In order to calculate the derivative, it is necessary to calculate $\frac{d}{d\theta}\left(\frac{\partial N_1}{\partial x}\right)$, $\frac{d}{d\theta}|\Omega_\theta|$. Topological derivatives can be calculated as

$$\frac{d}{d\Omega(x)}\left(\frac{dN_1}{dx}\right) = \frac{\frac{d}{d\theta}\left(\frac{dN_1}{dx}\right)}{\frac{d}{d\theta}|\Omega_\theta|} \tag{40}$$

In a similar way, it is possible to calculate the topological derivative of all elements of stiffness matrix. Above described topological derivative can be used to the calculations of extreme values of the set dependent functions and in the modeling of uncertainty. Let us consider the 2D plane stress FEM model from the Fig. 1 where P=1000 [N], L_x=L_y=1, $E = 2 \cdot 10^{12} \left[\frac{N}{m^2}\right]$, $\nu = 0.2$, h=0.1 [m] (thickness). Let us consider per-

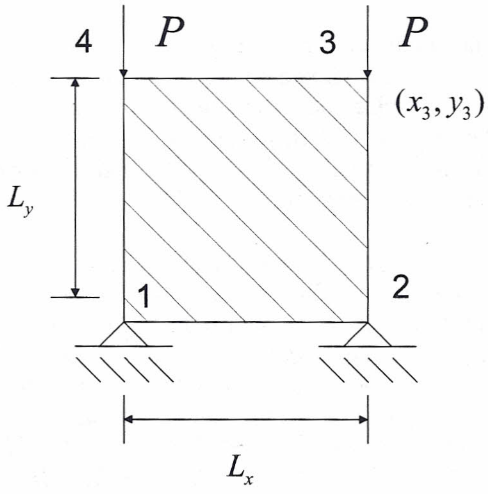

Figure 1: 2D FEM problem

turbation of the region in the direction of y axis. Let us consider y displacement of the node 3 in the y direction ($u_y^{(3)}$).The topological derivative $\frac{du_y^{(3)}}{d\Omega}$ can be calculated in the following way

$$\frac{du_y^{(3)}}{d\Omega} = \frac{\frac{du_y^{(3)}}{d\theta}}{\frac{d|\Omega_\theta|}{d\theta}} \tag{41}$$

The derivative $\frac{du_y^{(3)}}{d\theta}$ can be calculated form the implicit function theorem.

$$K\frac{du}{d\theta} = \frac{dQ}{d\theta} - \frac{dK}{d\theta}u \tag{42}$$

After calculations, we will get

$$\frac{du_y^{(3)}}{d\theta} = -9.91947 \cdot 10^{-9}. \tag{43}$$

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Then topological derivative

$$\frac{du_y^{(3)}}{d\Omega} = \frac{\frac{du_y^{(3)}}{d\theta}}{\frac{d|\Omega|}{d\theta}} = -9.91947 \cdot 10^{-9} \tag{44}$$

is negative, because of that

$$\Omega^{min} = \overline{\Omega}, \quad \Omega^{max} = \underline{\Omega} \tag{45}$$

then

$$\underline{u}_y^{(3)} = u_y^{(3)}(\Omega^{min}), \quad \overline{u}_y^{(3)} = u_y^{(3)}(\Omega^{max}). \tag{46}$$

For $\Delta\theta = 0.1[m]$ extreme values of the displacements are the following.

$$u_y^{(3)} \in [-1.0861 \cdot 10^{-8}, -8.87705 \cdot 10^{-9}][m] \tag{47}$$

The results confirm the intuition that if the region is higher (y coordinate grows), then the absolution value of the displacement in the y direction grows. Because the sign of that displacement is negative, then the function actually is decreasing and the topological derivative is negative.

Now let us consider a model which is shown in the Fig. 2. In calculations, the following numerical data is considered, Young's modulus $E \in [1.98 \cdot 10^{11}, 2.02 \cdot 10^{11}]$ $[Pa]$, Poisson ration $\nu \in [0.198, 0.202]$ the uncertain load $P \in [-1010, -990]$ $[N]$, and uncertain x coordinates of the supports $\Delta x = 0.01$ $[m]$ ($x_1 \in [-\Delta x, \Delta x]$, $x_3 \in [2L - \Delta x, 2L + \Delta x]$), L=10 $[m]$, $H = L = 1$ $[m]$, thickness w=0.01 $[m]$. In calculations 6 rectangular FEM elements were applied. Interval von Mises

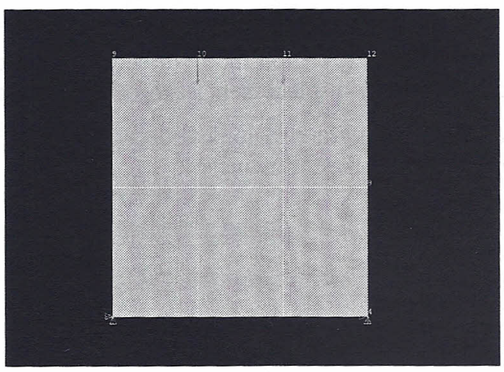

Figure 2: FEM model in ANSYS

stress are shown on the Fig. 3. Maximum von Mises stress is shown on the Fig. 4. Calculation was done by using special gradient free optimization method [11]. The appropriate software can be downloaded from

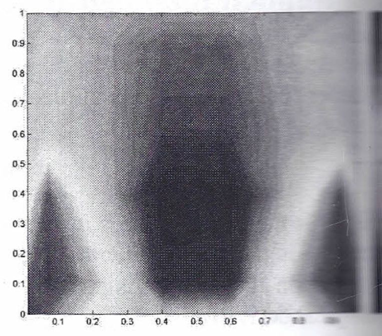

Figure 3: Interval von Mises

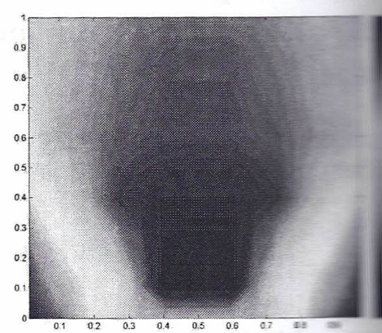

Figure 4: Maximum von Mises

the authors web page http://andrzej
On the same web page, it is possible
applications which automatically
for the calculations. The program
in C++ language and can be run
Linux.

9 Truss with uncertain geometry

Let us consider 11 bar truss [10], on the Fig. 5 with the uncertain $E \in [1.98 \cdot 10^{11}, 2.02 \cdot 10^{11}]$ $[Pa]$, $P \in [-15150, -14850]$ $[N]$, and dinates of the nodes 1 and 3 $\Delta x =$ $[-\Delta x, \Delta x]$, $x_3 \in [2L - \Delta x, 2L + \Delta x]$), $H = 5$ $[m]$, area of cross-section Interval displacements are shown in the

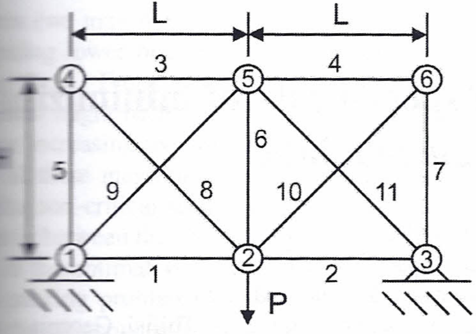

Figure 5: 11 bar truss

Table [illegible]: Interval displacement of the truss with uncertain geometry

	Lower bound [m]	Upper bound [m]
[illegible]	-2.883822e-005	2.886680e-005
[illegible]	-1.526831e-002	-1.463698e-002
[illegible]	-7.216225e-005	7.214392e-005
[illegible]	-1.296921e-002	-1.239865e-002

Conclusions

The presented concept of topological derivative can be applied in the efficient and large scale HPC computation. The equation (33) can be used in the framework of FEM, FVM or BEM method. The algorithm of modeling of uncertainty is the same as in the case of interval [1] and functional parameters [2]. The method is general, can be applied to the modeling a wide variety of problems in computational science. On the basis of the theory which was presented in this paper a general interval FEM program, which will be able to analyse uncertainty of problems with interval, functional interval and set interval parameters will be created. That will be a topic of future research. A general interval FEM program which is uses interval parameters can be downloaded from the the authors web page (http://andrzej.pownuk.com). Several working examples which are related to the Interval Finite Element are also presented on that web page.

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